14
Coadjoint Orbits

In this chapter we prove, amongst other things, that *the coadjoint orbits of a Lie group are symplectic manifolds*. These symplectic manifolds are, in fact, the symplectic leaves for the Lie–Poisson bracket. This result was developed and used by Kirillov, Arnold, Kostant, and Souriau in the early to mid-1960s, although it had important roots going back to the work of Lie, Borel, and Weil. (See Kirillov [1962, 1976b], Arnold [1966a], Kostant [1970], and Souriau [1969].) Here we give a direct proof. In Volume II we shall see a more "natural" proof using reduction.

Recall from Chapter 9 that the ***adjoint representation*** of a Lie group G is defined by

$$\mathrm{Ad}_g = T_e I_g : \mathfrak{g} \to \mathfrak{g},$$

where $I_g : G \to G$ is the inner automorphism $I_g(h) = ghg^{-1}$. The ***coadjoint action*** is given by

$$\mathrm{Ad}^*_{g^{-1}} : \mathfrak{g}^* \to \mathfrak{g}^*,$$

where $\mathrm{Ad}^*_{g^{-1}}$ is the dual of the linear map $\mathrm{Ad}_{g^{-1}}$, that is, it is defined by

$$\langle \mathrm{Ad}^*_{g^{-1}}(\mu), \xi \rangle = \langle \mu, \mathrm{Ad}_{g^{-1}}(\xi) \rangle,$$

where $\mu \in \mathfrak{g}^*, \xi \in \mathfrak{g}$, and $\langle\, , \rangle$ denotes the pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$. The ***coadjoint orbit***, $\mathrm{Orb}(\mu)$, through $\mu \in \mathfrak{g}^*$ is the subset of $\mathfrak{g}^*$ defined by

$$\mathrm{Orb}(\mu) := \{\mathrm{Ad}^*_{g^{-1}}(\mu) \mid g \in G\} := G \cdot \mu.$$

Like the orbit of any group action, $\mathrm{Orb}(\mu)$ is an *immersed submanifold* of $\mathfrak{g}^*$ and *if G is compact,* $\mathrm{Orb}(\mu)$ *is a closed embedded submanifold.*

14.1 Examples of Coadjoint Orbits

(a) Rotation Group. As we saw in §9.3, the adjoint action for $\mathrm{SO}(3)$ is

$$\mathrm{Ad}_{\mathrm{A}}(\mathbf{v}) = \mathbf{A}\mathbf{v}, \quad \text{where } \mathbf{A} \in \mathrm{SO}(3) \text{ and } \mathbf{v} \in \mathbb{R}^3 \cong \mathfrak{so}(3).$$

Identify $\mathfrak{so}(3)^*$ with $\mathbb{R}^3$ by the usual dot product, that is, if $\mathbf{\Pi}, \mathbf{v} \in \mathbb{R}^3$, we have $\langle \mathbf{\Pi}, \hat{\mathbf{v}} \rangle = \mathbf{\Pi} \cdot \mathbf{v}$. Thus, for $\mathbf{\Pi} \in \mathfrak{so}(3)^*$ and $\mathbf{A} \in \mathrm{SO}(3)$,

$$\begin{aligned}\langle \mathrm{Ad}^*_{\mathrm{A}^{-1}}(\mathbf{\Pi}), \hat{\mathbf{v}} \rangle &= \langle \mathbf{\Pi}, \mathrm{Ad}_{\mathrm{A}^{-1}}(\hat{\mathbf{v}}) \rangle = \langle \mathbf{\Pi}, (\mathbf{A}^{-1}\hat{\mathbf{v}}) \rangle = \mathbf{\Pi} \cdot \mathbf{A}^{-1}\mathbf{v} \\ &= (\mathbf{A}^{-1})^T \mathbf{\Pi} \cdot \mathbf{v} = \mathbf{A}\mathbf{\Pi} \cdot \mathbf{v} \end{aligned} \tag{14.1.1}$$

since $\mathbf{A}$ is orthogonal. Hence, with $\mathfrak{so}(3)^*$ identified with $\mathbb{R}^3, \mathrm{Ad}^*_{\mathrm{A}^{-1}} = \mathbf{A}$, and so

$$\mathrm{Orb}(\mathbf{\Pi}) = \{\mathrm{Ad}^*_{\mathrm{A}^{-1}}(\mathbf{\Pi}) \mid \mathbf{A} \in \mathrm{SO}(3)\} = \{\mathbf{A}\mathbf{\Pi} \mid \mathbf{A} \in \mathrm{SO}(3)\}, \tag{14.1.2}$$

which is the sphere in $\mathbb{R}^3$ of radius $\|\mathbf{\Pi}\|$. ♦

(b) Affine Group on $\mathbb{R}$. Consider the Lie group of transformations of $\mathbb{R}$ of the form $T(x) = ax + b$ where $a \neq 0$. Identify G with the set of pairs $(a, b) \in \mathbb{R}^2$ with $a \neq 0$. Since

$$(T_1 \circ T_2)(x) = a_1(a_2 x + b_2) + b_1 = a_1 a_2 x + a_1 b_2 + b_1$$

and

$$T^{-1}(x) = \frac{1}{a}(x - b),$$

we take group multiplication to be

$$(a_1, b_1) \cdot (a_2, b_2) = (a_1 a_2, a_1 b_2 + b_1). \tag{14.1.3}$$

The inverse of (a, b) is

$$(a, b)^{-1} = \left(\frac{1}{a}, -\frac{b}{a}\right) \tag{14.1.4}$$

and the identity element is $(1, 0)$. Thus, G is a two-dimensional Lie group. It is an example of a *semidirect product*. (See Exercise 9.3-1.) As a set, the Lie algebra of G is $\mathfrak{g} = \mathbb{R}^2$; to compute the bracket on $\mathfrak{g}$ we shall first compute the adjoint representation. The inner automorphisms are given by

$$\begin{aligned} I_{(a,b)}(c, d) &= (a, b) \cdot (c, d) \cdot (a, b)^{-1} \\ &= (ac, ad + b) \cdot \left(\frac{1}{a}, -\frac{b}{a}\right) \\ &= (c, ad - bc + b), \end{aligned} \tag{14.1.5}$$

and so differentiating (14.1.5) with respect to (c,d) at the identity in the direction of $(u,v) \in \mathfrak{g}$, gives

$$\mathrm{Ad}_{(a,b)}(u,v) = (u, av - bu). \tag{14.1.6}$$

Differentiating (14.1.6) with respect to (a,b) in the direction (r,s) gives the Lie bracket

$$[(r,s),(u,v)] = (0, rv - su). \tag{14.1.7}$$

The adjoint orbit through (u,v) is $\{u\} \times \mathbb{R}$ if $(u,v) \neq (0,0)$ and is $\{(0,0)\}$ if $(u,v) = (0,0)$. The *adjoint orbit* $\{u\} \times \mathbb{R}$ cannot be symplectic, as it is one dimensional. To compute the *coadjoint orbits*, denote elements of $\mathfrak{g}^*$ by $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ and the pairing

$$\left\langle (u,v), \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \right\rangle = \alpha u + \beta v \tag{14.1.8}$$

identifies $\mathfrak{g}^*$ with $\mathbb{R}^2$. Then

$$\begin{aligned} \left\langle \mathrm{Ad}^*_{(a,b)} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, (u,v) \right\rangle &= \left\langle \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \mathrm{Ad}_{(a,b)}(u,v) \right\rangle \\ &= \left\langle \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, (u, av - bu) \right\rangle \\ &= \alpha u + \beta a v - \beta b u. \end{aligned} \tag{14.1.9}$$

Thus,

$$\mathrm{Ad}^*_{(a,b)} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha - \beta b \\ \beta a \end{pmatrix}. \tag{14.1.10}$$

If $\beta = 0$, the coadjoint orbit through $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ is a single point. If $\beta \neq 0$, the orbit through $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ is $\mathbb{R}^2$ minus the β-axis. ♦

(c) Orbits in $\mathfrak{X}^*_{\mathrm{div}}$. Let $G = \mathrm{Diff}_{\mathrm{vol}}(\Omega)$, the group of volume-preserving diffeomorphisms of a region Ω in $\mathbb{R}^n$, with Lie algebra $\mathfrak{X}_{\mathrm{div}}(\Omega)$. In Example (d) of §10.2 we identified $\mathfrak{X}^*_{\mathrm{div}}(\Omega)$ with $\mathfrak{X}_{\mathrm{div}}(\Omega)$ by using the L^2-pairing on vector fields. Here we begin by finding a different representative of the dual $\mathfrak{X}^*_{\mathrm{div}}(\Omega)$, which is more convenient for explicitly determining the coadjoint action. Then we return to the identification above and will find the expression for the coadjoint action on $\mathfrak{X}_{\mathrm{div}}(\Omega)$; it will turn out to be more complicated.

The main technical ingredient used below is the Hodge decomposition theorem for manifolds with boundary. Here we state only the relevant facts to be used below. A k-form α is said to be ***tangent*** to $\partial\Omega$ if $i^*(*\alpha) = 0$. Let $\Omega_t^k(\Omega)$ denote all k-forms on M which are tangent to $\partial\Omega$. One of the Hodge decomposition theorems states that there is an L^2-orthogonal decomposition

$$\Omega^k(\Omega) = \mathbf{d}\Omega^{k-1}(\Omega) \oplus \{\alpha \in \Omega_t^k(\Omega) \mid \delta\alpha = 0\}.$$

This implies that the pairing

$$\langle\,,\rangle : \{\alpha \in \Omega_t^1(\Omega) \mid \delta\alpha = 0\} \times \mathfrak{X}_{\mathrm{div}}(\Omega) \to \mathbb{R}$$

given by

$$\langle M, X\rangle = \int_\Omega M_i X^i d^n x. \tag{14.1.11}$$

is weakly nondegenerate. Indeed, if

$$M \in \{\alpha \in \Omega_t^1(M) \mid \delta\alpha = 0\}$$

and $\langle M, X\rangle = 0$ for all $X \in \mathfrak{X}_{\mathrm{div}}(\Omega)$, then $\langle M, B\rangle = 0$ for all

$$B \in \{\Omega_t^1(\Omega) \mid \delta B = 0\}$$

because the index lowering operator $^\flat$ given by the metric on Ω induces an ismorphism between $\mathfrak{X}_{\mathrm{div}}(\Omega)$ and

$$\{\alpha \in \Omega_t^1(\Omega) \mid \delta B = 0\}.$$

Therefore, by the L^2-orthogonal decomposition quoted above, $M = \mathbf{d}f$ and hence $M = 0$. Similarly, if $X \in \mathfrak{X}_{\mathrm{div}}(\Omega)$ and $\langle M, X\rangle = 0$ for all $M \in \{\alpha \in \Omega_t^1(M) \mid \delta\alpha = 0\}$, then $\langle M, X^\flat\rangle = 0$ for all such M, and as before $X^\flat = \mathbf{d}f$, that is, $X = \nabla f$. But this implies $X = 0$ since $\mathfrak{X}_{\mathrm{div}}(\Omega)$ and gradients are L^2-orthogonal by the Stokes theorem. Therefore, we can identify

$$\mathfrak{X}_{\mathrm{div}}^*(\Omega) = \{M \in \Omega_t^1(\Omega) \mid \delta M = 0\}. \tag{14.1.12}$$

The coadjoint action of $\mathrm{Diff}_{\mathrm{vol}}(\Omega)$ on $\mathfrak{X}_{\mathrm{div}}^*(\Omega)$ is computed in the following way. Recall from Chapter 9 that $\mathrm{Ad}_\varphi(X) = \varphi_* X$ for $\varphi \in \mathrm{Diff}_{\mathrm{vol}}(\Omega)$ and $X \in \mathfrak{X}_{\mathrm{div}}(\Omega)$. Thus,

$$\langle \mathrm{Ad}_{\varphi^{-1}}^* M, X\rangle = \langle M, \mathrm{Ad}_{\varphi^{-1}} X\rangle = \int_\Omega M \cdot \varphi^* X\, d^n x = \int_\Omega \varphi_* M \cdot X d^n x$$

by the change of variables formula. Therefore,

$$\mathrm{Ad}_{\varphi^{-1}}^* M = \varphi_* M \quad \text{and so} \quad \mathrm{Orb}\, M = \{\varphi_* M \mid \varphi \in \mathrm{Diff}_{\mathrm{vol}}(\Omega)\}. \tag{14.1.13}$$

Next, let us return to the identification of $\mathfrak{X}_{\mathrm{div}}(\Omega)$ with itself by the L^2-pairing on vector fields

$$\langle X, Y\rangle = \int_\Omega X \cdot Y \, d^n x. \tag{14.1.14}$$

The Helmholtz decomposition says that any vector field on Ω can be uniquely decomposed orthogonally in a sum of a gradient of a function and a divergence-free vector field tangent to $\partial\Omega$; this decomposition is equivalent to the Hodge decomposition on one-forms quoted before. This shows that (14.1.14) is a weakly nondegenerate pairing. For $\varphi \in \mathrm{Diff}_{\mathrm{vol}}(\Omega)$, denote by $(T\varphi)^\dagger$ the adjoint of $T\varphi : T\Omega \to T\Omega$ relative to the metric (14.1.14). By the change of variables formula,

$$\begin{aligned}
\langle \mathrm{Ad}^*_{\varphi^{-1}} Y, X\rangle = \langle Y, \mathrm{Ad}_{\varphi^{-1}} X\rangle &= \int_\Omega Y \cdot \varphi^* X \, d^n x \\
&= \int_\Omega Y \cdot (T\varphi^{-1} \circ X \circ \varphi)\, d^n x \\
&= \int_\Omega ((T\varphi^{-1})^\dagger \circ Y \circ \varphi) \cdot X \; d^n x,
\end{aligned}$$

that is,

$$\mathrm{Ad}^*_{\varphi^{-1}} Y = (T\varphi^{-1})^\dagger \circ Y \circ \varphi \tag{14.1.15}$$

and

$$\mathrm{Orb}\, Y = \{(T\varphi^{-1})^\dagger \circ Y \circ \varphi \mid \varphi \in \mathrm{Diff}_{\mathrm{vol}}(\Omega)\}. \tag{14.1.16}$$

This example shows that different pairings give rise to different formulae for the coadjoint action and that the choice of dual is dictated by the specific application one has in mind. For example, the pairing (14.1.14) was convenient for the Lie–Poisson bracket on $\mathfrak{X}_{\mathrm{div}}(\Omega)$ in Example (d) of §10.2. On the other hand, many computations involving the coadjoint action are simpler with the choice (14.1.12) of the dual corresponding to the pairing (14.1.11). ♦

(d) Orbits in $\mathfrak{X}^*_{\mathrm{can}}$. Let $G = \mathrm{Diff}_{\mathrm{can}}(P)$ be the group of canonical transformations of a symplectic manifold P with $H^1(P) = 0$. Letting k be a function on P, and X_k the corresponding Hamiltonian vector field, and $\varphi \in G$, we have

$$\mathrm{Ad}_\varphi X_k = \varphi_* X_k = X_{k\circ\varphi^{-1}} \tag{14.1.17}$$

so identifying $\mathfrak{g}$ with $\mathcal{F}(P)$ modulo constants, or equivalently with functions on P with zero average, we get $\mathrm{Ad}_\varphi \, k = \varphi_* k = k \circ \varphi^{-1}$. On the dual space, which is identified with $\mathcal{F}(P)$ (modulo constants) via the L^2-pairing, a straightforward verification shows that

$$\mathrm{Ad}^*_{\varphi^{-1}} f = \varphi_* f = f \circ \varphi^{-1}. \tag{14.1.18}$$

One sometimes says that

$$\mathrm{Orb}(f) = \{ f \circ \varphi^{-1} \mid \varphi \in \mathrm{Diff}_{\mathrm{can}}(P) \}$$

consists of ***canonical rearrangements*** of f. ♦

(e) Toda Orbit. Another interesting example is the Toda orbit, which arises in the study of completely integrable systems. Let

$\mathfrak{g}$ = Lie algebra of real $n \times n$ lower triangular matrices of trace zero,

G = lower triangular matrices with determinant one,

and identify

$$\mathfrak{g}^* = \text{ the upper triangular matrices,}$$

using the pairing

$$\langle \xi, \mu \rangle = \mathrm{Trace}(\xi \mu),$$

where $\xi \in \mathfrak{g}$ and $\mu \in \mathfrak{g}^*$. Since $\mathrm{Ad}_A \xi = A \xi A^{-1}$, we get

$$\mathrm{Ad}^*_{A^{-1}} \mu = P(A \mu A^{-1}), \tag{14.1.19}$$

where $P : \mathfrak{sl}(n, \mathbb{R}) \to \mathfrak{g}^*$ is the projection sending any matrix to its upper triangular part. Now let

$$\mu = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \in \mathfrak{g}^*. \tag{14.1.20}$$

One finds that $\mathrm{Orb}(\mu) = \{ P(A \mu \, A^{-1}) \mid A \in G \}$ consists of matrices of the form

$$L = \begin{bmatrix} b_1 & a_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & b_2 & a_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & b_3 & a_3 & \cdots & 0 & 0 \\ 0 & 0 & 0 & b_4 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & b_{n-1} & a_{n-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & b_n \end{bmatrix}, \tag{14.1.21}$$

where $\sum b_n = 0$. See Kostant [1980] and Symes [1982a,b] for further information. ♦

(f) Coadjoint Orbits That Are Not Submanifolds. The following example of a Lie group G, whose generic coadjoint orbits in $\mathfrak{g}^*$ are *not* submanifolds, is due to Kirillov [1976b], p. 293. Let α be irrational, define

$$G = \left\{ \left[\begin{array}{ccc} e^{it} & 0 & z \\ 0 & e^{i\alpha t} & w \\ 0 & 0 & 1 \end{array} \right] \middle| \, t \in \mathbb{R}, z, w \in \mathbb{C} \right\}, \tag{14.1.22}$$

and note the G is diffeomorphic to $\mathbb{R}^5$. As a group it is the semidirect product of

$$H = \left\{ \left[\begin{array}{cc} e^{it} & 0 \\ 0 & e^{i\alpha t} \end{array} \right] \middle| \, t \in \mathbb{R} \right\}$$

with $\mathbb{C}^2$, the action being by left multiplication of vectors in $\mathbb{C}^2$ by elements of H (see Exercise 9.3-1). The Lie algebra $\mathfrak{g}$ of G is

$$\mathfrak{g} = \left\{ \left[\begin{array}{ccc} it & 0 & x \\ 0 & i\alpha t & y \\ 0 & 0 & 0 \end{array} \right] \middle| \, t \in \mathbb{R}, x, y \in \mathbb{C} \right\} \tag{14.1.23}$$

with the usual commutator bracket as Lie bracket. Identify $\mathfrak{g}^*$ with

$$\mathfrak{g}^* = \left\{ \left[\begin{array}{ccc} is & 0 & 0 \\ 0 & i\alpha s & 0 \\ a & b & 0 \end{array} \right] \middle| \, s \in \mathbb{R}, a, b \in \mathbb{C} \right\} \tag{14.1.24}$$

via the nondegenerate pairing in $\mathfrak{gl}(3, \mathbb{C})$ is given by

$$\langle A, B \rangle = \operatorname{Re}\left(\operatorname{trace}(AB)\right).$$

The adjoint action of

$$g = \left[\begin{array}{ccc} e^{it} & 0 & z \\ 0 & e^{i\alpha t} & w \\ 0 & 0 & 1 \end{array} \right] \quad \text{on} \quad \xi = \left[\begin{array}{ccc} is & 0 & x \\ 0 & i\alpha s & y \\ 0 & 0 & 0 \end{array} \right]$$

is given by

$$\operatorname{Ad}_g \xi = \left[\begin{array}{ccc} is & 0 & e^{it}x - isz \\ 0 & i\alpha s & e^{i\alpha t}y - i\alpha s w \\ 0 & 0 & 0 \end{array} \right]. \tag{14.1.25}$$

The coadjoint action of the same group element g on

$$\mu = \left[\begin{array}{ccc} iu & 0 & 0 \\ 0 & i\alpha u & 0 \\ a & b & 0 \end{array} \right]$$

is given by

$$\operatorname{Ad}^*_{g^{-1}} \mu = \begin{bmatrix} iu' & 0 & 0 \\ 0 & i\alpha u' & 0 \\ ae^{-it} & be^{-i\alpha t} & 0 \end{bmatrix}, \tag{14.1.26}$$

where

$$u' = u + \frac{1}{1+\alpha^2}\operatorname{Im}(ae^{-it}z + be^{-i\alpha t}\alpha w). \tag{14.1.27}$$

If $a, b \neq 0$, the orbit through μ is two dimensional; it is a cylindrical surface whose generator is the u'-axis and whose base is the curve in $\mathbb{C}^2$ given parametrically by $t \mapsto (ae^{-it}, be^{-i\alpha t})$. This curve, however, is the irrational flow on the torus with radii $|a|$ and $|b|$, that is, the cylindrical surface accumulates on itself and thus is not a submanifold of $\mathbb{R}^5$. We shall return to this example at the end of §14.6. ♦

14.2 Tangent Vectors to Coadjoint Orbits

In general, orbits of a Lie group action, while manifolds in their own right, are not submanifolds of the ambient manifold; they are only injectively immersed manifolds. A notable exception occurs in the case of compact Lie groups: then all their orbits are embedded submanifolds. Coadjoint orbits are no exception to this global problem, as we saw in the preceding examples. We shall always regard them as injectively immersed submanifolds, diffeomorphic to G/G_μ, where $G_\mu = \{g \in G \mid \operatorname{Ad}^*_g \mu = \mu\}$ is the isotropy subgroup of the coadjoint action at a point μ in the orbit.

We now describe tangent vectors to coadjoint orbits. Let $\xi \in \mathfrak{g}$ and let $g(t)$ be a curve in G tangent to ξ at $t = 0$; for example, let $g(t) = \exp(t\xi)$. Let $\mathcal{O}$ be a coadjoint orbit, and $\mu \in \mathcal{O}$. If $\eta \in \mathfrak{g}$, then

$$\mu(t) = \operatorname{Ad}^*_{g(t)^{-1}}\mu \tag{14.2.1}$$

is a curve in $\mathcal{O}$ with $\mu(0) = \mu$. Differentiating the identity

$$\langle \mu(t), \eta\rangle = \langle \mu, \operatorname{Ad}_{g(t)^{-1}}\eta\rangle \tag{14.2.2}$$

with respect to t at $t = 0$, we get

$$\langle \mu'(0), \eta\rangle = -\langle \mu, \operatorname{ad}_\xi \eta\rangle = -\langle \operatorname{ad}^*_\xi \mu, \eta\rangle, \quad \text{and so} \quad \mu'(0) = -\operatorname{ad}^*_\xi \mu. \tag{14.2.3}$$

Thus,

$$T_\mu\mathcal{O} = \{\operatorname{ad}^*_\xi \mu \mid \xi \in \mathfrak{g}\}. \tag{14.2.4}$$

This calculation also proves that the infinitessimal generator of the coadjoint action is given by

$$\xi_{\mathfrak{g}^*}(\mu) = -\operatorname{ad}^*_\xi \mu. \tag{14.2.5}$$

The following characterization of the tangent space to coadjoint orbits is often useful. We let $\mathfrak{g}_\mu = \{\xi \in \mathfrak{g} \mid \operatorname{ad}^*_\xi \mu = 0\}$ be the coadjoint isotropy algebra of μ; it is the Lie algebra of the coadjoint isotropy group $G_\mu = \{g \in G \mid \operatorname{Ad}^*_g \mu = \mu\}$.

Proposition 14.2.1. *Let $\langle\, ,\rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ be a weakly nondegenerate pairing and let $\mathcal{O}$ be the coadjoint orbit through $\mu \in \mathfrak{g}^*$. Let*

$$\mathfrak{g}_\mu^\circ := \{\nu \in \mathfrak{g}^* \mid \langle \nu, \eta\rangle = 0 \quad \textit{for all } \eta \in \mathfrak{g}_\mu\}$$

be the ***annihilator*** *of $\mathfrak{g}_\mu$ in $\mathfrak{g}^*$. Then $T_\mu\mathcal{O} \subset \mathfrak{g}_\mu^\circ$. If $\mathfrak{g}$ is finite dimensional, then $T_\mu\mathcal{O} = \mathfrak{g}_\mu^\circ$. The same equality holds if $\mathfrak{g}$ and $\mathfrak{g}^*$ are Banach spaces, $T_\mu\mathcal{O}$ is closed in $\mathfrak{g}^*$, and the pairing is strongly nondegenerate.*

Proof. For any $\xi \in \mathfrak{g}, \eta \in \mathfrak{g}_\mu$ we have

$$\langle \operatorname{ad}^*_\xi \mu, \eta\rangle = \langle \mu, [\xi, \eta]\rangle = -\langle \operatorname{ad}^*_\eta \mu, \xi\rangle = 0,$$

which proves the inclusion $T_\mu\mathcal{O} \subset \mathfrak{g}_\mu^\circ$. If $\mathfrak{g}$ is finite dimensional, equality holds since $\dim T_\mu\mathcal{O} = \dim \mathfrak{g} - \dim \mathfrak{g}_\mu = \dim \mathfrak{g}_\mu^\circ$. If $\mathfrak{g}$ and $\mathfrak{g}^*$ are infinite-dimensional Banach spaces and $\langle\, ,\rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ is a strong pairing, we can assume without loss of generality that it is the natural pairing between a Banach space and its dual. If $\mathfrak{g}_\mu^\circ \neq T_\mu\mathcal{O}$ pick $\nu \neq 0, \nu \in \mathfrak{g}_\mu^\circ, \nu \notin T_\mu\mathcal{O}$. By the Hahn–Banach theorem there is an $\eta \in \mathfrak{g}$ such that $\langle \nu, \eta\rangle = 1$ and $\langle \operatorname{ad}^*_\xi \mu, \eta\rangle = 0$ for all $\xi \in \mathfrak{g}$. The latter condition is equivalent to $\eta \in \mathfrak{g}_\mu$. On the other hand, since $\nu \in \mathfrak{g}_\mu^\circ$ we have $\langle \nu, \eta\rangle = 0$, which is a contradiction. ■

Examples of Tangent Vectors

(a) Rotation Group. Identifying $(\mathfrak{so}(3), [\cdot\,,\cdot]) \cong (\mathbb{R}^3, \times)$ and $\mathfrak{so}(3)^* \cong \mathbb{R}^3$ via the natural pairing given by the Euclidean inner product, formula (14.2.5) reads as follows for $\mathbf{\Pi} \in \mathfrak{so}(3)^*$ and $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathfrak{so}(3)$,

$$\langle \boldsymbol{\xi}_{\mathfrak{so}(3)^*}(\mathbf{\Pi}), \boldsymbol{\eta}\rangle = -\mathbf{\Pi} \cdot (\boldsymbol{\xi} \times \boldsymbol{\eta}) = -(\mathbf{\Pi} \times \boldsymbol{\xi}) \cdot \boldsymbol{\eta} \tag{14.2.6}$$

so that $\boldsymbol{\xi}_{\mathfrak{so}(3)^*}(\mathbf{\Pi}) = -\mathbf{\Pi} \times \boldsymbol{\xi} = \boldsymbol{\xi} \times \mathbf{\Pi}$. As expected, $\boldsymbol{\xi}_{\mathfrak{so}(3)^*}(\mathbf{\Pi}) \in T_\Pi \operatorname{Orb}(\mathbf{\Pi})$ is tangent to the sphere $\operatorname{Orb}(\mathbf{\Pi})$. Allowing $\boldsymbol{\xi}$ to vary in $\mathfrak{so}(3) \cong \mathbb{R}^3$, one obtains all of $T_\Pi \operatorname{Orb}(\mathbf{\Pi})$. ♦

(b) Affine Group on $\mathbb{R}$. Let $(u, v) \in \mathfrak{g}$ and consider the coadjoint orbit through the point $\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in \mathfrak{g}^*$. Then (14.2.5) reads

$$(u, v)_{\mathfrak{g}^*} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \left\langle \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, [\,\cdot\,, (u, v)] \right\rangle. \tag{14.2.7}$$

But $\left\langle \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, [(r, s), (u, v)] \right\rangle = \left\langle \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, (0, rv - su) \right\rangle = rv\beta - su\beta$, and so

$$(u, v)_{\mathfrak{g}^*} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} v\beta \\ -u\beta \end{pmatrix}. \tag{14.2.8}$$

If $\beta \neq 0$, these vectors span $\mathfrak{g}^* = \mathbb{R}^2$ as they should. ♦

(c) The Group $\mathrm{Diff}_{\mathrm{vol}}$. For $G = \mathrm{Diff}_{\mathrm{vol}}$ and $M \in \mathfrak{X}^*_{\mathrm{div}}$, we get the tangent vectors to $\mathrm{Orb}(M)$ by differentiating (14.1.13) with respect to φ, yielding

$$T_M \ \mathrm{Orb}(M) = \{-\pounds_v M \mid v \text{ is divergence free and tangent to } \partial\Omega\}. \tag{14.2.9}$$

♦

(d) The Group $\mathbf{Diff_{can}}\,(\mathbf{P})$. For $G = \mathrm{Diff}_{\mathrm{can}}(P)$, we have

$$T_f \ \mathrm{Orb}(f) = \{-\{f, k\} \mid k \in \mathcal{F}(P)\}. \tag{14.2.10}$$

♦

(e) The Toda Lattice. The tangent space to the Toda orbit consists of matrices of the same form as L in (14.1.21) since those matrices form a linear space. The reader can check that (14.2.4) gives the same answer. ♦

14.3 The Symplectic Structure on Coadjoint Orbits

Theorem 14.3.1 (Coadjoint Orbit Theorem). *Let G be a Lie group and let $\mathcal{O} \subset \mathfrak{g}^*$ be a coadjoint orbit. Then $\mathcal{O}$ is a symplectic manifold. In fact, there are unique symplectic forms $\omega^{\pm}$ on $\mathcal{O}$ such that*

$$\omega^{\pm}(\mu)(\xi_{\mathfrak{g}^*}(\mu), \eta_{\mathfrak{g}^*}(\mu)) = \pm\langle \mu, [\xi, \eta]\rangle \tag{14.3.1}$$

for all $\mu \in \mathcal{O}$ and $\xi, \eta \in \mathfrak{g}$. We refer to $\omega^{\pm}$ as the ***coadjoint orbit symplectic structures*** *and, if there is danger of confusion, denote it $\omega^{\pm}_{\mathcal{O}}$.*

Proof. We prove the result for ω^-, the argument for ω^+ being similar. First we show that formula (14.3.1) gives a well-defined form; that is, the right-hand side is independent of the particular $\xi \in \mathfrak{g}$ and $\eta \in \mathfrak{g}$ which define the tangent vectors $\xi_{\mathfrak{g}^*}(\mu)$ and $\eta_{\mathfrak{g}^*}(\mu)$. This follows by observing that $\xi_{\mathfrak{g}^*}(\mu) = \xi'_{\mathfrak{g}^*}(\mu)$ implies $-\langle \mu, [\xi, \eta] \rangle = -\langle \mu, [\xi', \eta] \rangle$ for all $\eta \in \mathfrak{g}$. Therefore, $\omega^-(\mu)(\xi_{\mathfrak{g}^*}(\mu), \eta_{\mathfrak{g}^*}(\mu))\rangle = \omega^-(\xi'_{\mathfrak{g}^*}(\mu), \eta_{\mathfrak{g}^*}(\mu))\rangle$, so ω^- is well defined.

Second, we show that ω^- is nondegenerate. Since the pairing $\langle \, , \rangle$ is nondegenerate, $\omega^-(\mu)(\xi_{\mathfrak{g}^*}(\mu), \eta_{\mathfrak{g}^*}(\mu)) = 0$ for all $\eta_{\mathfrak{g}^*}(\mu)$ implies $-\langle \mu, [\xi, \eta] \rangle = 0$ for all η. This means that $0 = -\langle \mu, [\xi, \cdot] \rangle = \xi_{\mathfrak{g}^*}(\mu)$.

Finally, we show that ω^- is closed, that is $\mathbf{d}\omega^- = 0$. To do this we begin by defining, for each $\nu \in \mathfrak{g}^*$, the one-form ν_L on G by

$$\nu_L(g) = (T_g^* L_{g^{-1}})(\nu),$$

where $g \in G$. The one-form ν_L is readily checked to be left invariant; that is $L_g^* \nu_L = \nu_L$ for all $g \in G$. For $\xi \in \mathfrak{g}$, let ξ_L be the corresponding left invariant vector field on G, so $\nu_L(\xi_L)$ is a constant function on G (whose value at any point is $\langle \nu, \xi \rangle$). Choose $\nu \in \mathcal{O}$ and consider the surjective map $\varphi_\nu : G \to \mathcal{O}$ defined by $g \mapsto \mathrm{Ad}^*_{g^{-1}}(\nu)$ and the two-form $\sigma = \varphi_\nu^* \omega^-$ on G. We claim that

$$\sigma = \mathbf{d}\nu_L. \tag{14.3.2}$$

To prove this, notice that

$$(T_e \varphi_\nu)(\eta) = \eta_{\mathfrak{g}^*}(\nu) \tag{14.3.3}$$

so that the surjective map φ_ν is submersive at e. By definition of pull back, $\sigma(e)(\xi, \eta)$ equals

$$\begin{aligned}(\varphi_\nu^* \omega^-)(e)(\xi, \eta) &= \omega^-(\varphi_\nu(e))(T_e\varphi_\nu \cdot \xi, T_e\varphi_\nu \cdot \eta) \\ &= \omega^-(\nu)(\xi_{\mathfrak{g}^*}(\nu), \eta_{\mathfrak{g}^*}(\nu)) = -\langle \nu, [\xi, \eta] \rangle. \end{aligned} \tag{14.3.4}$$

Hence

$$\sigma(\xi_L, \eta_L)(e) = \sigma(e)(\xi, \eta) = -\langle \nu, [\xi, \eta] \rangle = -\langle \nu_L, [\xi_L, \eta_L] \rangle (e). \tag{14.3.5}$$

We shall need the relation $\sigma(\xi_L, \eta_L) = -\langle \nu_L, [\xi_L, \eta_L] \rangle$ at each point of G; to get it, we first prove two lemmas. ▼

Lemma 14.3.2. $\mathrm{Ad}^*_{g^{-1}} : \mathcal{O} \to \mathcal{O}$ *preserves* ω^-*, that is,* $(\mathrm{Ad}^*_{g^{-1}})^* \omega^- = \omega^-$.

Is ▼ correct as end-of-proof symbols for the following lemmas? (wendy)

Proof. To prove this, we recall two identities from Chapter 9. First,

$$(\mathrm{Ad}_g \, \xi)_{\mathfrak{g}^*} = \mathrm{Ad}^*_{g^{-1}} \circ \xi_{\mathfrak{g}^*} \circ \mathrm{Ad}^*_g, \tag{14.3.6}$$

which is proved by letting ξ be tangent to a curve $h(\varepsilon)$ at $\varepsilon = 0$, recalling that

$$\mathrm{Ad}_g\, \xi = \frac{d}{d\varepsilon} g h(\varepsilon) g^{-1}\Big|_{\varepsilon=0} \tag{14.3.7}$$

and noting

$$\begin{aligned}(\mathrm{Ad}_g\, \xi)_{\mathfrak{g}^*}(\mu) &= \frac{d}{d\varepsilon}\, \mathrm{Ad}^*_{(gh(\varepsilon)g^{-1})^{-1}}\, \mu\Big|_{\varepsilon=0} \\ &= \frac{d}{d\varepsilon}\, \mathrm{Ad}^*_{g^{-1}}\, \mathrm{Ad}^*_{h(\varepsilon)^{-1}}\, \mathrm{Ad}^*_g(\mu)\Big|_{\varepsilon=0}. \end{aligned} \tag{14.3.8}$$

Second, we require the identity

$$\mathrm{Ad}_g[\xi, \eta] = [\mathrm{Ad}_g\, \xi, \mathrm{Ad}_g\, \eta], \tag{14.3.9}$$

which follows by differentiating the relation

$$I_g(I_h(k)) = I_g(h) I_g(k) I_g(h^{-1}) \tag{14.3.10}$$

with respect to h and k and evaluating at the identity.

Evaluating (14.3.6) at $\nu = \mathrm{Ad}^*_{g^{-1}}\, \mu$, we get

$$(\mathrm{Ad}_g\, \xi)_{\mathfrak{g}^*}(\nu) = \mathrm{Ad}^*_{g^{-1}} \cdot \xi_{\mathfrak{g}^*}(\mu) = T_\mu\, \mathrm{Ad}^*_{g^{-1}} \cdot \xi_{\mathfrak{g}^*}(\mu), \tag{14.3.11}$$

by linearity of $\mathrm{Ad}^*_{g^{-1}}$. Thus,

$$\begin{aligned}&((\mathrm{Ad}^*_{g^{-1}})^* \omega^-)(\mu)(\xi_{\mathfrak{g}^*}(\mu), \eta_{\mathfrak{g}^*}(\mu)) \\ &\quad = \omega^-(\nu)(T_\mu\, \mathrm{Ad}^*_{\mathfrak{g}^{-1}} \cdot \xi_{\mathfrak{g}^*}(\mu), T_\mu\, \mathrm{Ad}^*_{\mathfrak{g}^{-1}} \cdot \eta_{\mathfrak{g}^*}(\mu)) \\ &\quad = \omega^-(\nu)((\mathrm{Ad}_g\, \xi)_{\mathfrak{g}^*}(\nu), (\mathrm{Ad}_g\, \eta)_{g^*}(\nu)) && \text{(by (14.3.11))} \\ &\quad = -\langle \nu, [\mathrm{Ad}_g\, \xi, \mathrm{Ad}_g\, \eta]\rangle && \text{(by definition of } \omega^-) \\ &\quad = -\langle \nu, \mathrm{Ad}_g[\xi, \eta]\rangle && \text{(by (14.3.9))} \\ &\quad = -\left\langle \mathrm{Ad}^*_g\, \nu, [\xi, \eta]\right\rangle = -\langle \mu, [\xi, \eta]\rangle \\ &\quad = \omega^-(\mu)(\xi_{\mathfrak{g}^*}(\mu), \eta_{\mathfrak{g}^*}(\mu)). \end{aligned} \tag{14.3.12}$$

▼

Lemma 14.3.3. *σ is left invariant, that is, $L_g^* \sigma = \sigma$ for all $g \in G$.*

Proof. Using the equivariance identity $\varphi_\nu \circ L_g = \mathrm{Ad}^*_{g^{-1}} \circ \varphi_\nu$, we compute

$$\begin{aligned}L_g^* \sigma &= L_g^* \varphi_\nu^* \omega^- = (\varphi_\nu \circ L_g)^* \omega^- = (\mathrm{Ad}^*_{g^{-1}} \circ \varphi_\nu)^* \omega^- \\ &= \varphi_\nu^* (\mathrm{Ad}^*_{g^{-1}})^* \omega^- = \varphi_\nu^* \omega^- = \sigma. \end{aligned}$$

▼

Lemma 14.3.4. $\sigma(\xi_L, \eta_L) = -\langle \nu_L, [\xi_L, \eta_L] \rangle$.

Proof. Both sides are left invariant and are equal at the identity by (14.3.5). ▼

The exterior derivative $\mathbf{d}\alpha$ of a one-form α is given in terms of the Jacobi–Lie bracket by

$$(\mathbf{d}\alpha)(X, Y) = X[\alpha(Y)] - Y[\alpha(X)] - \alpha([X, Y]). \tag{14.3.13}$$

Since $\nu_L(\xi_L)$ is constant, $\eta_L[\nu_L(\xi_L)] = 0$ and $\xi_L[\nu_L(\eta_L)] = 0$, so Lemma 14.4.4 implies

$$\sigma(\xi_L, \eta_L) = (\mathbf{d}\nu_L)(\xi_L, \eta_L). \tag{14.3.14}$$

Lemma 14.3.5.

$$\sigma = \mathbf{d}\nu_L. \tag{14.3.15}$$

Proof. We shall prove that for any vector fields X and Y, $\sigma(X, Y) = (\mathbf{d}\nu_L)(X, Y)$. Indeed, since σ is left invariant,

$$\begin{aligned}
\sigma(X, Y)(g) &= (L^*_{g^{-1}}\sigma)(g)(X(g), Y(g)) \\
&= \sigma(e)(TL_{g^{-1}} \cdot X(g), TL_{g^{-1}} \cdot Y(g)) \\
&= \sigma(e)(\xi, \eta) \quad (\text{where } \xi = TL_{g^{-1}} \cdot X(g) \text{ and } \eta = TL_{g^{-1}} \cdot Y(g)) \\
&= \sigma(\xi_L, \eta_L)(e) = (\mathbf{d}\nu_L)(\xi_L, \eta_L)(e) \qquad (\text{by } (14.3.14)) \\
&= (L^*_g \mathbf{d}\nu_L)(\xi_L, \eta_L)(e) \qquad (\text{since } \nu_L \text{ is } left \text{ invariant}) \\
&= (\mathbf{d}\nu_L)(g)(TL_g \cdot \xi_L(e), TL_g \cdot \eta_L(e)) \\
&= (\mathbf{d}\nu_L)(g)(TL_g \cdot \xi, TL_g \cdot \eta) = (\mathbf{d}\nu_L)(g)(X(g),\ Y(g)) \\
&= (\mathbf{d}\nu_L)(X, Y)(g).
\end{aligned}$$
▼

Since $\sigma = \mathbf{d}\nu_L$ by Lemma 14.4.5, $\mathbf{d}\sigma = \mathbf{dd}\nu_L = 0$, and so $0 = \mathbf{d}\varphi_\nu^*\omega^- = \varphi_\nu^*\mathbf{d}\omega^-$. From $\varphi_\nu \circ L_g = \mathrm{Ad}^*_{g^{-1}} \circ \varphi_\nu$, it follows that submersivity of φ_ν at e is equivalent to submersivity of φ_ν at any $g \in G$, that is, φ_ν is a surjective submersion. Thus, φ_ν^* is injective, and hence $\mathbf{d}\omega^- = 0$.

Remark. Any Lie group carries a natural connection associated to the left (or right) action. The calculation (14.3.13) is essentially the calculation of the curvature of this connection and, as such, is closely related to the *Maurer–Cartan equations* (see §9.1). ♦

Since coadjoint orbits are symplectic, we get the following:

Corollary 14.3.6. *Coadjoint orbits of finite-dimensional Lie groups are even dimensional.*

Corollary 14.3.7. *Let $G_\nu = \{g \in G \mid \mathrm{Ad}^*_{g^{-1}}\, \nu = \nu\}$ be the isotropy subgroup of the coadjoint action of $\nu \in \mathfrak{g}^*$. Then G_ν is a closed subgroup of G, and so the quotient G/G_ν is a smooth manifold with smooth projection $\pi : G \to G/G_\nu; g \mapsto g \cdot G_\nu$. We identify $G/G_\nu \cong \mathrm{Orb}(\nu)$ via the diffeomorphism $\rho : g \cdot G_\nu \in G/G_\nu \mapsto \mathrm{Ad}^*_{g^{-1}}(\nu) \in \mathrm{Orb}(\nu)$. Thus, G/G_ν is symplectic, with symplectic form ω^- induced from $\mathbf{d}\nu_L$, that is,*

$$\mathbf{d}\nu_L = \pi^* \rho^* \omega^-$$

(*respectively,* $\mathbf{d}\nu_R = \pi^*\rho^*\omega^+$).

As we shall see in Example (a) of §14.5, ω^- is not exact in general, even though $\pi^*\rho^*\omega^-$ is.

Examples

(a) Rotation Group. Consider $\mathrm{Orb}(\boldsymbol{\Pi})$, the coadjoint orbit through $\boldsymbol{\Pi} \in \mathbb{R}^3$; then

$$\boldsymbol{\xi}_{\mathbb{R}^3}(\boldsymbol{\Pi}) = \boldsymbol{\xi} \times \boldsymbol{\Pi} \in T_\Pi(\mathrm{Orb}(\boldsymbol{\Pi})), \text{ and } \boldsymbol{\eta}_{\mathbb{R}^3}(\boldsymbol{\Pi}) = \boldsymbol{\eta} \times \boldsymbol{\Pi} \in T_\Pi(\mathrm{Orb}(\boldsymbol{\Pi})),$$

and so with the usual identification of $\mathfrak{so}(3)$ with $\mathbb{R}^3$, the (–) coadjoint orbit symplectic structure becomes

$$\omega^-(\boldsymbol{\xi}_{\mathbb{R}^3}(\boldsymbol{\Pi}), \boldsymbol{\eta}_{\mathbb{R}^3}(\boldsymbol{\Pi})) = -\boldsymbol{\Pi} \cdot (\boldsymbol{\xi} \times \boldsymbol{\eta}). \tag{14.3.16}$$

Recall that the oriented area of the (planar) parallelogram spanned by two vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$, is given by $\mathbf{v} \times \mathbf{w}$ (the numerical area is $\|\mathbf{v} \times \mathbf{w}\|$). Thus, the oriented area spanned by $\boldsymbol{\xi}_{\mathbb{R}^3}(\boldsymbol{\Pi})$ and $\boldsymbol{\eta}_{\mathbb{R}^3}(\boldsymbol{\Pi})$ is $(\boldsymbol{\xi} \times \boldsymbol{\Pi}) \times (\boldsymbol{\eta} \times \boldsymbol{\Pi}) = [(\boldsymbol{\xi} \times \boldsymbol{\Pi}) \cdot \boldsymbol{\Pi}]\boldsymbol{\eta} - [(\boldsymbol{\xi} \times \boldsymbol{\Pi}) \cdot \boldsymbol{\eta}]\boldsymbol{\Pi} = \boldsymbol{\Pi}(\boldsymbol{\Pi} \cdot (\boldsymbol{\xi} \times \boldsymbol{\eta}))$.

The area element dA on a sphere in $\mathbb{R}^3$ assigns to each pair $(\mathbf{v}, \mathbf{w})$ of tangent vectors the number $dA(\mathbf{v}, \mathbf{w}) = \mathbf{n} \cdot (\mathbf{v} \times \mathbf{w})$, where $\mathbf{n}$ is the unit outward normal (this is the area of the parallelogram spanned by $\mathbf{v}$ and $\mathbf{w}$, taken "+" if $\mathbf{v}, \mathbf{w}, \mathbf{n}$ form a positively oriented basis and "−" otherwise). For a sphere of radius $\|\boldsymbol{\Pi}\|$ and tangent vectors $\mathbf{v} = \boldsymbol{\xi} \times \boldsymbol{\Pi}$ and $\mathbf{w} = \boldsymbol{\eta} \times \boldsymbol{\Pi}$, we have

$$\begin{aligned} dA(\boldsymbol{\xi} \times \boldsymbol{\Pi}, \boldsymbol{\eta} \times \boldsymbol{\Pi}) &= \frac{\boldsymbol{\Pi}}{\|\boldsymbol{\Pi}\|} \cdot ((\boldsymbol{\xi} \times \boldsymbol{\Pi}) \times (\boldsymbol{\eta} \times \boldsymbol{\Pi})) \\ &= \frac{\boldsymbol{\Pi}}{\|\boldsymbol{\Pi}\|} \cdot ((\boldsymbol{\xi} \times \boldsymbol{\Pi}) \cdot \boldsymbol{\Pi})\boldsymbol{\eta} - ((\boldsymbol{\xi} \times \boldsymbol{\Pi}) \cdot \boldsymbol{\eta})\boldsymbol{\Pi}) \\ &= \|\boldsymbol{\Pi}\| \boldsymbol{\Pi} \cdot (\boldsymbol{\xi} \times \boldsymbol{\eta}). \end{aligned} \tag{14.3.17}$$

Thus,

$$\omega^- = -\frac{1}{\|\boldsymbol{\Pi}\|} dA. \tag{14.3.18}$$

The use of "dA" for the area element is, of course, a notational abuse since this two-form cannot be exact. Likewise,

$$\omega^+ = \frac{1}{\|\mathbf{\Pi}\|} dA. \tag{14.3.19}$$

Notice that $\omega^+/\|\mathbf{\Pi}\| = (dA)/\|\mathbf{\Pi}\|^2$ is the solid angle subtended by the area element dA. ♦

(b) Affine Group on $\mathbb{R}$. For $\beta \neq 0$, and $\mu = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ on the open orbit $\mathcal{O}$, formula (14.3.1) gives

$$\begin{aligned} \omega^-(\mu)((r,s)_{\mathfrak{g}^*}(\mu), (u,v)_{\mathfrak{g}^*}(\mu) &= -\left\langle \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, [(r,s),(u,v)] \right\rangle \\ &= \beta(rv - su), \end{aligned} \tag{14.3.20}$$

or in coordinates, $(q,p) \in \mathbb{R}^2$,

$$\omega^-(\mu) = \beta \mathbf{d}q \wedge \mathbf{d}p. \tag{14.3.21}$$

♦

(c) The Group $\mathrm{Diff}_{\mathrm{vol}}$. For a coadjoint orbit of $G = \mathrm{Diff}_{\mathrm{vol}}(\Omega)$ the $(+)$ coadjoint orbit symplectic structure at a point M becomes

$$\omega^+(M)(-\pounds_v M, -\pounds_w M) = -\int_\Omega M \cdot [v,w]\, d^n x, \tag{14.3.22}$$

where $[v,w]$ is the Jacobi–Lie bracket. Note that we have indeed a minus sign on the right-hand side of (14.3.22) since $[v,w]$ is *minus* the left Lie algebra bracket. ♦

Exercises

◇ **Exercise 14.3-1.** Let G be a Lie group. Find an action of G on T^*G for which the map $J(\xi)(\nu_L(g)) = -\langle \nu_L(g), \xi_L(g)\rangle = -\langle \nu, \xi\rangle$ is an equivariant momentum map.

◇ **Exercise 14.3-2.** Relate the calculations of this section to the Mauer-Cartan equations.

◇ **Exercise 14.3-3.** Give another proof that $\mathbf{d}\omega^\pm = 0$ by showing that X_H for $\omega^\pm$ coincides with that for the Lie–Poisson bracket and hence that Jacobi's identity holds.

◇ **Exercise 14.3-4 (The Group $\mathrm{Diff}_{\mathrm{can}}$).** For a coadjoint orbit for $G = \mathrm{Diff}_{\mathrm{can}}(P)$, show that the $(+)$ coadjoint orbit symplectic structure is

$$\omega^+(f)(\{k,f\},\{h,f\}) = \int_P f\{k,h\}\, dq\, dp.$$

◇ **Exercise 14.3-5 (The Toda Lattice).** For the Toda orbit, check that the orbit symplectic structure is

$$\omega^{+}(f) = \sum_{i=1}^{n-1} \frac{1}{a_i}\, \mathbf{d}b_i \wedge \mathbf{d}a_i. \tag{14.3.23}$$

14.4 The Orbit Bracket via Restriction of the Lie–Poisson Bracket

Theorem 14.4.1 (Lie–Poisson-Coadjoint Orbit Compatibility). *The Lie–Poisson bracket and the coadjoint orbit symplectic structure are consistent in the following sense: for $F, H : \mathfrak{g}^* \to \mathbb{R}$ and $\mathcal{O}$ a coadjoint orbit in $\mathfrak{g}^*$,*

$$\{F, H\}_+ | \mathcal{O} = \{F|\mathcal{O}, H|\mathcal{O}\}^+. \tag{14.4.1}$$

Here, the bracket $\{F, G\}_+$ is the $(+)$ Lie–Poisson bracket, while the bracket on the right-hand side of (14.4.1) *is the Poisson bracket defined by the* $(+)$ *coadjoint orbit symplectic structure on $\mathcal{O}$. Similarly,*

$$\{F, H\}_- | \mathcal{O} = \{F|\mathcal{O}, H|\mathcal{O}\}^-. \tag{14.4.2}$$

The following box summarizes the basic content of what the theorem says.

Two Approaches to the Lie–Poisson Bracket

There are two different ways to produce the same Lie–Poisson bracket $\{F, H\}_-$ (respectively, $\{F, H\}_+$) on $\mathfrak{g}^*$:

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Extension Method:

1. Take $F, H : \mathfrak{g}^* \to \mathbb{R}$;
2. extend F, H to $F_L, H_L : T^*G \to \mathbb{R}$ by left (respectively, right) invariance;
3. take the bracket $\{F_L, H_L\}$ with respect to the canonical symplectic structure on T^*G; and
4. restrict: $\{F_L, H_L\}|\mathfrak{g}^* = \{F, H\}_-$ (respectively, $\{F_R, H_R\}|\mathfrak{g}^* = \{F, H\}_+$).

overflow!

Restriction Method:

1. Take $F, H : \mathfrak{g}^* \to \mathbb{R}$;
2. form the restrictions $F|\mathcal{O}, H|\mathcal{O}$ to a coadjoint orbit; and
3. take the Poisson bracket $\{F|\mathcal{O}, H|\mathcal{O}\}^-$ with respect to the $-$ (respectively, $+$) orbit symplectic structure ω^- (respectively, ω^+) on the orbit $\mathcal{O}$: for $\mu \in \mathcal{O}$ we have

$$\{F|\mathcal{O}, H|\mathcal{O}\}^-(\mu) = \{F, H\}_-(\mu).$$

Proof of Theorem 14.6.1. Let $\mu \in \mathcal{O}$. By definition,

$$\{F, H\}_-(\mu) = -\left\langle \mu, \left[\frac{\delta F}{\delta \mu}, \frac{\delta H}{\delta \mu}\right]\right\rangle. \tag{14.4.3}$$

On the other hand,

$$\{F|\mathcal{O}, H|\mathcal{O}\}^-(\mu) = \omega^-(X_F, X_H)(\mu), \tag{14.4.4}$$

where X_F and X_H are the Hamiltonian vector fields on $\mathcal{O}$ generated by $F|\mathcal{O}$ and $H|\mathcal{O}$, and ω^- is the minus orbit symplectic form. Recall that the Hamiltonian vector field X_F on $\mathfrak{g}^*_-$ is given by

$$X_F(\mu) = \mathrm{ad}^*_\xi(\mu), \tag{14.4.5}$$

where $\xi = \delta F/\delta\mu \in \mathfrak{g}$.

Motivated by this we prove the following:

Lemma 14.4.2. *Using the orbit symplectic form ω^-, for $\mu \in \mathcal{O}$ we have*

$$X_{F|\mathcal{O}}(\mu) = \mathrm{ad}^*_{\delta F/\delta\mu}(\mu). \tag{14.4.6}$$

■

Proof. Let $\xi, \eta \in \mathfrak{g}$, so (14.3.1) gives

$$\omega^-(\mu)(\mathrm{ad}^*_\xi\, \mu, \mathrm{ad}^*_\eta\, \mu) = -\langle \mu, [\xi, \eta]\rangle = \langle \mu, \mathrm{ad}_\eta(\xi)\rangle = \langle \mathrm{ad}^*_\eta(\mu), \xi\rangle. \tag{14.4.7}$$

Letting $\xi = \delta F/\delta\mu$ and η be arbitrary, we get

$$\omega^-(\mu)(\mathrm{ad}^*_{\delta F/\delta\mu}\, \mu, \mathrm{ad}^*_\eta\, \mu) = \left\langle \mathrm{ad}^*_\eta\, \mu, \frac{\delta F}{\delta\mu}\right\rangle = \mathbf{d}F(\mu) \cdot \mathrm{ad}^*_\eta\, \mu. \tag{14.4.8}$$

Thus, $X_{F|\mathcal{O}}(\mu) = \mathrm{ad}^*_{\delta F/\delta\mu}\, \mu$, as required. ▼

To complete the proof of Theorem 14.6.1, note that

$$\begin{aligned}\{F|\mathcal{O}, H|\mathcal{O}\}^-(\mu) &= \omega^-(\mu)(X_{F|\mathcal{O}}(\mu), X_{H|\mathcal{O}}(\mu)) \\ &= \omega^-(\mu)(\mathrm{ad}^*_{\delta F/\delta\mu}\,\mu, \mathrm{ad}^*_{\delta H/\delta\mu}\,\mu) \\ &= -\left\langle \mu, \left[\frac{\delta F}{\delta\mu}, \frac{\delta H}{\delta\mu}\right]\right\rangle = \{F, H\}_-(\mu). \qquad (14.4.9)\end{aligned}$$

check this Corollary

Corollary 14.4.3. (i) *For $H \in \mathcal{F}(\mathfrak{g}^*)$, the trajectory of X_H starting at μ stays in* $\mathrm{Orb}(\mu)$.

(ii) *A function $C \in \mathcal{F}(\mathfrak{g}^*)$ is a Casimir iff $\delta C/\delta\mu \in \mathfrak{g}_\mu$ for all $\mu \in \mathfrak{g}^*$.*

(iii) *If $C \in \mathcal{F}(\mathfrak{g}^*)$ is* Ad^**-invariant (constant on orbits) then C is a Casimir. The converse is also true if all coadjoint orbits are connected.*

Proof.
Part (i) follows from the fact that $X_H(\nu)$ is tangent to the coadjoint orbit $\mathcal{O}$ for $\nu \in \mathcal{O}$, since $X_H(\nu) = \mathrm{ad}^*_{\delta H/\delta\mu}(\nu)$. Part (ii) follows from the definitions and formula (14.4.5), and (iii) follows from (ii) by writing out the condition of Ad^*-invariance as $C(\mathrm{Ad}^*_{g^{-1}}\,\mu) = C(\mu)$ and differentiating in g at $g = e$.

The converse is proved in the following way. If P is a Poisson manifold, $S \subset P$ is a symplectic leaf, and C is a Casimir function, then C is necessarily constant on S. Indeed, if C were not locally a constant on S, then there would be a point $z \in S$ such that $\mathbf{d}C(z) \cdot v \neq 0$ for some $v \in T_zS$. But T_zS is spanned by $X_k(z)$ for k varying over $\mathcal{F}(P)$ and hence $\mathbf{d}C(z) \cdot X_k(z) = \{C, k\}(z) = 0$. Therefore, $\mathbf{d}C(z) \cdot v = 0$, a contradiction. Thus, C is locally constant on S and hence constant, by connectedness of the leaf S. In particular, if all coadjoint orbits of a Lie group G in $\mathfrak{g}^*$ are connected, then a Casimir function C is constant on each orbit and hence Ad^*-invariant. ■

To illustrate part (iii), we note that for $G = \mathrm{SO}(3)$, the function

$$C_\Phi(\mathbf{\Pi}) = \Phi\left(\frac{1}{2}\|\mathbf{\Pi}\|^2\right)$$

is invariant under the coadjoint action $(\mathbf{A}, \mathbf{\Pi}) \mapsto \mathbf{A\Pi}$ and is therefore a Casimir function. Another example is given by $G = \mathrm{Diff}_{\mathrm{can}}(P)$, and the functional

$$C_\Phi(f) := \int_P \Phi(f)\, dq\, dp,$$

where $dq\,dp$ is the Liouville measure and Φ is any function of one variable. This is a Casimir since it is Ad^*-invariant by the change of variables formula.

In general, Ad^*-invariance of C is a stronger condition than C being a Casimir function. Indeed if C is Ad^*-invariant, differentiating the relation $C(\mathrm{Ad}^*_{g^{-1}}\,\mu) = C(\mu)$ relative to μ rather than g as we did in the proof of (iii), we get

$$\frac{\delta C}{\delta(\mathrm{Ad}^*_{g^{-1}}\,\mu)} = \mathrm{Ad}_g\,\frac{\delta C}{\delta\mu} \tag{14.4.10}$$

for all $g \in G$. Taking $g \in G_\mu$, this relation becomes $\delta C/\delta\mu = \mathrm{Ad}_g(\delta C/\delta\mu)$, that is, $\delta C/\delta\mu$ belongs to the centralizer of G_μ in $\mathfrak{g}$, that is, to the set

$$\mathrm{Cent}(G_\mu, \mathfrak{g}) := \{\xi \in \mathfrak{g} \mid \mathrm{Ad}_g\,\xi = \xi \quad \text{for all } g \in G_\mu\}.$$

Letting

$$\mathrm{Cent}(\mathfrak{g}_\mu, \mathfrak{g}) := \{\xi \in \mathfrak{g} \mid [\eta, \xi] = 0 \quad \text{for all } \eta \in \mathfrak{g}_\mu\}$$

denote the centralizer of $\mathfrak{g}_\mu$ in $\mathfrak{g}$, we see by differentiating the relation defining $\mathrm{Cent}(G_\mu, \mathfrak{g})$ with respect to g at the identity, that $\mathrm{Cent}(G_\mu, \mathfrak{g}) \subset \mathrm{Cent}(\mathfrak{g}_\mu, \mathfrak{g})$. Thus, if C is Ad^*-invariant, then

$$\frac{\delta C}{\delta\mu} \in \mathfrak{g}_\mu \cap \mathrm{Cent}(\mathfrak{g}_\mu, \mathfrak{g}) = \mathrm{Cent}(\mathfrak{g}_\mu) = \text{ the center of } \mathfrak{g}_\mu.$$

Proposition 14.4.4 (Kostant [1979]). *If C is an Ad^*-invariant function on $\mathfrak{g}^*$, then $\delta C/\delta\mu$ lies in both $\mathrm{Cent}(G_\mu, \mathfrak{g})$ and in $\mathrm{Cent}(\mathfrak{g}_\mu)$. If C is a Casimir function, then $\delta C/\delta\mu$ lies in the center of $\mathfrak{g}_\mu$.*

Proof. The preceding calculations prove the first statement. The last statement follows from the observation that if C is a Casimir function for G, it is also one for G_0, the connected component of the identity, and so from XXX 14.6.3iii it is G_0-Ad^*-invariant, so the first statement applies. ■

By the theorem of Duflo and Vergne [1969] (see Chapter 9), for generic $\mu \in \mathfrak{g}^*$, the coadjoint isotropy $\mathfrak{g}_\mu$ is abelian and therefore $\mathrm{Cent}(\mathfrak{g}_\mu) = \mathfrak{g}_\mu$ generically. The above corollary and proposition leave open, in principle, the possibility of non-Ad^*-invariant Casimir functions on $\mathfrak{g}^*$. This is not possible for Lie groups with connected coadjoint orbits, as we saw before. It is also not possible for semisimple Lie groups since any Casimir function is a functional of the basis of the ring of invariants. *If $C : \mathfrak{g}^* \to \mathbb{R}$ is a function such that $\delta C/\delta\mu \in \mathfrak{g}_\mu$ for all $\mu \in \mathfrak{g}^*$, but there is at least one $\nu \in \mathfrak{g}^*$ such that $\delta C/\delta\nu \notin \mathrm{Cent}(\mathfrak{g}_\nu)$, then C is a Casimir function that is not* Ad^**-invariant.* This element $\nu \in \mathfrak{g}^*$ must be such that its coadjoint orbit is disconnected, it must be nongeneric, and $\mathfrak{g}$ must be non-semisimple. We know of no such example of a Casimir function.

On the other hand, the above statements provide easily verifiable criteria for the form or the nonexistence of Casimir functions on duals of Lie algebras. For example, if $\mathfrak{g}^*$ has open orbits whose union is dense, it cannot have Casimir functionals. Indeed, any Casimir would have to be constant on each orbit, and thus by continuity, on $\mathfrak{g}^*$. An example of such a Lie algebra is that of the affine group on the line discussed in Example (b) of §14.1. The same argument shows that Lie algebras with at least one dense orbit have no Casimir functionals.

Let us use Corollary 14.6.3 to determine all Casimir functions for the Lie algebra in Example (f) of §14.1. If

$$\mu = \begin{bmatrix} iu & 0 & 0 \\ 0 & i\alpha u & 0 \\ a & b & 0 \end{bmatrix} \in \mathfrak{g}^*, \quad \xi = \begin{bmatrix} is & 0 & x \\ 0 & i\alpha s & y \\ 0 & 0 & 0 \end{bmatrix} \in \mathfrak{g},$$

for $a, b, x, y \in \mathbb{C}, u, s \in \mathbb{R}$, then it is straightforward to check that

$$\operatorname{ad}^*_\xi \mu = \begin{bmatrix} iu'' & 0 & 0 \\ 0 & i\alpha u'' & 0 \\ -ixa & -i\alpha sb & 0 \end{bmatrix} \quad \text{for} \quad u'' = \frac{1}{1+\alpha^2}\operatorname{Im}(ax + \alpha bxy).$$

Thus, if at least one of a, b is not zero, then

$$\mathfrak{g}_\mu = \left\{ \begin{bmatrix} 0 & 0 & x \\ 0 & 0 & y \\ 0 & 0 & 0 \end{bmatrix} \middle| \operatorname{Im}(ax + \alpha by) = 0 \right\},$$

whereas if $a = b = 0$, then $\mathfrak{g}_\mu = \mathfrak{g}$. For $C : \mathfrak{g}^* \to \mathbb{R}$ denote by

$$\frac{\delta C}{\delta \mu} = \begin{bmatrix} iC_u & 0 & C_a \\ 0 & i\alpha C_u & C_b \\ 0 & 0 & 0 \end{bmatrix},$$

where $C_u \in \mathbb{R}, C_a, C_b \in \mathbb{C}$ are the partial derivatives of C relative to the variables u, a, b. Thus, the condition $\delta C/\delta\mu \in \mathfrak{g}_\mu$ for all μ implies that $C_u = 0$, that is, C is independent of u and

$$\operatorname{Im}(aC_a + \alpha b C_b) = 0.$$

The same condition could have been obtained by lengthier direct calculations involving the Lie–Poisson bracket. Here are the highlights. The commutator bracket on $\mathfrak{g}$ is given by

$$\left[\begin{bmatrix} is & 0 & x \\ 0 & i\alpha s & y \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} iu & 0 & z \\ 0 & i\alpha u & w \\ 0 & 0 & 0 \end{bmatrix} \right] = \begin{bmatrix} 0 & 0 & i(sz - ux) \\ 0 & 0 & i\alpha(sw - uy) \\ 0 & 0 & 0 \end{bmatrix},$$

so that for $\mu \in \mathfrak{g}^*$ parametrized by $u \in \mathbb{R}, a, b, \in \mathbb{C}$, we have

$$\begin{aligned} \{F, H\}_-(\mu) &= -\operatorname{Re}\operatorname{Tr}\left(\mu\left[\frac{\delta F}{\delta\mu}, \frac{\delta H}{\delta\mu}\right]\right) \\ &= \operatorname{Im}[a(F_u H_a - H_u F_a) + \alpha b(F_u H_b - H_u F_b)]. \end{aligned}$$

Taking $F_u = F_b = 0$ in $\{F, C\}_- = 0$, forces $C_u = 0$. Then the remaining condition reduces to $\operatorname{Im}(aC_a + \alpha b C_b) = 0$.

We solve this partial differential equation by the method of characteristics. Let $a = x + iy, b = u + iv$ so that $C_a = C_x + iC_y,\ C_b = C_u + iC_v$, and we get

$$xC_y + yC_x + \alpha v C_u + \alpha u C_v = 0.$$

The flow of the vector field with components $(y, x, \alpha v, \alpha u)$ is given by

$$F_t(x, y, u, v) = \Big(x \cosh t + y \sinh t,\ x \sinh t + y \cosh t,\\ u \cosh \alpha t + v \sinh \alpha t,\ u \sinh \alpha t + v \cosh \alpha t \Big)$$

and thus any function $C = f(x^2 - y^2, u^2 - v^2)$ is constant on this flow. Therefore, these functions are all Casimir functionals for $\mathfrak{g}^*$.

One mathematical reason coadjoint orbits and the Lie–Poisson bracket are so important is that every Hamiltonian space is (a covering of) a coadjoint orbit. This is proved below.

If X and Y are topological spaces, a continuous surjective map $p : X \to Y$ is called a ***covering map*** if every point in Y has an open neighborhood U such that $p^{-1}(U)$ is a disjoint union of open sets in X, called the ***decks*** over U. Note that each deck is homeomorphic to U by p. If $p : M \to N$ is a surjective proper map of smooth manifolds which is also a local diffeomorphism, then it is a covering map. For example, SU(2) (the spin group) forms a covering space of SO(3) with two decks over each point and SU(2) is simply connected while SO(3) is not. (See Chapter 9.)

Transitive Hamiltonian actions have been characterized by Lie, Kostant, Kirillov, and Souriau in the following manner (see Kostant [1966]):

Theorem 14.4.5 (Kostant's Coadjoint Orbit Covering Theorem). *Let P be a Poisson manifold and let $\Phi : G \times P \to P$ be a left, transitive, Hamiltonian action with equivariant momentum map $\mathbf{J} : P \to \mathfrak{g}^*$. Then*

(i) $\mathbf{J} : P \to \mathfrak{g}^*_+$ *is a canonical submersion onto a coadjoint orbit of G in* $\mathfrak{g}^*$.

(ii) *If P is symplectic,* $\mathbf{J}$ *is a symplectic local diffeomorphism onto a coadjoint orbit endowed with the "+" orbit symplectic structure. If* $\mathbf{J}$ *is also proper, then it is a covering map.*

Proof. (i) That $\mathbf{J}$ is a canonical map was proved in §12.5. Since Φ is transitive, choosing a $z_0 \in P$, any $z \in P$ can be written as $z = \Phi_g(z_0)$ for some $g \in G$. Thus, by equivariance

$$\begin{aligned}\mathbf{J}(P) = \{\mathbf{J}(z) \mid z \in P\} &= \{\mathbf{J}(\Phi_g(z_0)) \mid g \in G\} \\ &= \{\operatorname{Ad}^*_{g^{-1}} \mathbf{J}(z_0) \mid g \in G\} = \operatorname{Orb}(\mathbf{J}(z_0)).\end{aligned}$$

Again by equivariance, for $z \in P$ we have $T_z\mathbf{J}(\xi_P(z)) = -\operatorname{ad}^*_\xi \mathbf{J}(z)$, which has the form of a general tangent vector at $\mathbf{J}(z)$ to the orbit $\operatorname{Orb}(\mathbf{J}(z_0))$; thus, $\mathbf{J}$ is a submersion.

(ii) If P is symplectic with symplectic form $\Omega, \mathbf{J}$ is a symplectic map if the orbit has the "+" symplectic form: $\omega^+(\mu)(\operatorname{ad}^*_\xi \mu, \operatorname{ad}^*_\eta \mu) = \langle \mu, [\xi,\eta]\rangle$. This is seen in the following way. Since $T_zP = \{\xi_P(z) \mid \xi \in \mathfrak{g}\}$ by transitivity of the action,

$$\begin{aligned}
(\mathbf{J}^*\omega^+)(z)(\xi_P(z),\eta_P(z)) &= \omega^+(\mathbf{J}(z))(T_z\mathbf{J}(\xi_P(z)), T_z\mathbf{J}(\eta_P(z)))\\
&= \omega^+(\mathbf{J}(z))(\operatorname{ad}^*_\xi \mathbf{J}(z), \operatorname{ad}^*_\eta \mathbf{J}(z))\\
&= \langle \mathbf{J}(z), [\xi,\eta]\rangle = J([\xi,\eta])(z)\\
&= \{J(\xi), J(\eta)\}(z) \qquad \text{(by equivariance)}\\
&= \Omega(z)(X_{J(\xi)}(z), X_{J(\eta)}(z))\\
&= \Omega(z)(\xi_P(z),\eta_P(z)), \qquad (14.4.11)
\end{aligned}$$

which shows that $\mathbf{J}^*\omega^+ = \Omega$, that is, $\mathbf{J}$ is symplectic. Since any symplectic map is an immersion, $\mathbf{J}$ is a local diffeomorphism. If $\mathbf{J}$ is also proper, it is a symplectic covering map, as discussed above.

■

If $\mathbf{J}$ is proper and the symplectic manifold P is simply connected, the covering map in (ii) is a diffeomorphism; this follows from classical theorems about covering spaces (Spanier [1966]). It is clear that if Φ is not transitive, $\mathbf{J}(P)$ is a union of coadjoint orbits. See Guillemin and Sternberg [1984] and Grigore and Popp [1989] for more information.

Exercises

◇ **Exercise 14.4-1.** Show that if C is a Casimir function on a Poisson manifold, then $\{F,K\}_C = C\{F,K\}$ is also a Poisson structure.

14.5 The Special Linear Group on the Plane

Jerry to rewrite based on his class notes

In the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ of traceless real 2×2 matrices, introduce the basis

$$\mathbf{e} = \begin{bmatrix} 0 & 1\\ 0 & 0\end{bmatrix}, \quad \mathbf{f} = \begin{bmatrix} 0 & 0\\ 1 & 0\end{bmatrix}, \quad \mathbf{h} = \begin{bmatrix} 1 & 0\\ 0 & -1\end{bmatrix}.$$

Note that $[\mathbf{h},\mathbf{e}] = 2\mathbf{e}$, $[\mathbf{h},\mathbf{f}] = -2\mathbf{f}$, and $[\mathbf{e},\mathbf{f}] = \mathbf{h}$. Identify $\mathfrak{sl}(2,\mathbb{R})$ with $\mathbb{R}^3$ via

$$x\mathbf{e} + y\mathbf{f} + z\mathbf{h} \in \mathfrak{sl}(2,\mathbb{R}) \mapsto (x,y,z) \in \mathbb{R}^3. \qquad (14.5.1)$$

The nonzero structure constants are $c^3_{12} = 1, c^1_{13} = -2$, and $c^2_{23} = 2$. We identify the dual space $\mathfrak{sl}(2,\mathbb{R})^*$ with $\mathbb{R}^3$ via the map

$$\alpha \in \mathfrak{sl}(2,\mathbb{R})^* \mapsto (a,b,c) \in \mathbb{R}^3, \tag{14.5.2}$$

where $(a,b,c) \in \mathbb{R}^3$ is uniquely determined by the equality

$$\langle \alpha, x\mathbf{e} + y\mathbf{f} + z\mathbf{h}\rangle = ax + by + cz, \quad \text{i.e.,} \quad \alpha(\mathbf{e}) = a, \alpha(\mathbf{f}) = b, \alpha(\mathbf{h}) = c. \tag{14.5.3}$$

One calculates that the $(\pm)$ Lie–Poisson bracket of $\mathfrak{sl}(2,\mathbb{R})^*$ induces the following Poisson brackets on $\mathbb{R}^3$: $\{c,a\} = 2a$ (respectively, $-2a$), $\{c,b\} = -2b$ (respectively, $+2b$), $\{a,b\} = c$ (respectively, $-c$), that is,

$$\begin{aligned}\{F,G\}_\pm(a,b,c) = \mp 2a\left(\frac{\partial F}{\partial a}\frac{\partial G}{\partial c} - \frac{\partial F}{\partial c}\frac{\partial G}{\partial a}\right) \pm \ 2b\left(\frac{\partial F}{\partial b}\frac{\partial G}{\partial c} - \frac{\partial F}{\partial c}\frac{\partial G}{\partial b}\right)\\ \pm c\left(\frac{\partial F}{\partial a}\frac{\partial G}{\partial b} - \frac{\partial F}{\partial b}\frac{\partial G}{\partial a}\right).\end{aligned} \tag{14.5.4}$$

Any Casimir function of $\mathbb{R}^3$ endowed with the $(\pm)$ Lie–Poisson bracket of $\mathfrak{sl}(2,\mathbb{R})^*$ is of the form

$$C(a,b,c) = \Phi\left(ab + \frac{1}{4}c^2\right) \tag{14.5.5}$$

for a C^1 function $\Phi : \mathbb{R} \to \mathbb{R}$. Thus, the symplectic leaves are the sheets of the hyperboloids

$$C_0(a,b,c) := \frac{1}{2}\left(ab + \frac{1}{4}c^2\right) = \text{constant} \neq 0, \tag{14.5.6}$$

the two nappes (without vertex) of the cone $ab + (1/4)c^2 = 0$, and the origin. The orbit symplectic structure on these hyperboloids is given by

$$\begin{aligned}&\omega^-(a,b,c)(\mathrm{ad}^*_{(x,y,z)}(a,b,c), \mathrm{ad}^*_{(x',y',z')}(a,b,c))\\ &\quad = -a(2zx' - 2xz') - b(2yz' - 2zy') - c(xy' - yx')\\ &\quad = -\frac{1}{\|\nabla C_0(a,b,c)\|}\text{ (area element of the hyperboloid).}\end{aligned} \tag{14.5.7}$$

To prove the last equality in (14.5.7), use the formulae

$$\begin{aligned}&\mathrm{ad}^*_{(x,y,z)}(a,b,c) = (2az - cy, cx - 2bz, 2by - 2zx),\\ &\qquad\qquad \mathrm{ad}^*_{(x,y,z)}(a,b,c) \times \mathrm{ad}^*_{(x',y',z')}(a,b,c)\\ &\qquad = \big(2bc(xy' - yx') + 4b^2(yz' - zy') + 4ab(zx' - xz'),\\ &\qquad\quad 2ac(xy' - yx') + 4ab(yz' - zy') + 4a^2(zx' - xz'),\\ &\qquad\quad c^2(xy' - yx') + 2bc(yz' - zy') + 2ac(zx' - xz')\big),\end{aligned}$$

and the fact that $\nabla(ab + \frac{1}{4}c^2) = (b, a, \frac{1}{2}c)$ is normal to the hyperboloid to get, as in (14.3.18),

$$\begin{aligned}
&dA(a,b,c)(\mathrm{ad}^*_{(x,y,z)}(a,b,c), \mathrm{ad}^*_{(x',y',z')}(a,b,c)) \\
&\quad = \frac{(b,a,\frac{1}{2}c)}{\|(b,a,\frac{1}{2}c)\|} \cdot (\mathrm{ad}^*_{(x,y,z)}(a,b,c) \times \mathrm{ad}^*_{(x',y',z')}(a,b,c)) \\
&\quad = -\|\nabla C_0(a,b,c)\|)\omega^-(a,b,c)(\mathrm{ad}^*_{(x,y,z)}(a,b,c), \mathrm{ad}^*_{(x',y',z')}(a,b,c)).
\end{aligned}$$

14.6 The Euclidean Group of the Plane

We use the notation and terminology from Exercise 11.5-2. Recall that $\mathfrak{se}(2)^*$ is isomorphic to $\mathbb{R}^3$ with the bracket.

$$\begin{aligned}
[(\omega, v_1, v_2), (\zeta, w_1, w_2)] &= (0, \zeta v_2 - \omega w_2, \omega w_1 - \zeta v_1) \\
&= (0, \omega \mathbb{J}^T \mathbf{w} - \zeta \mathbb{J}^T \mathbf{v}),
\end{aligned} \tag{14.6.1}$$

where $\mathbf{v} = (v_1, v_2), \mathbf{w} = (w_1, w_2)$ and

$$\mathbb{J} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \mathbb{J}^t = \mathbb{J}^{-1} = -\mathbb{J}. \tag{14.6.2}$$

Thus, $\mathfrak{se}(2)^*$ is identified with $\mathbb{R}^3$ via the dot product. Therefore, if $F : \mathfrak{se}(2)^* \cong \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$, its functional derivative is

$$\frac{\delta F}{\delta(\mu, \alpha)} = \left(\frac{\partial F}{\partial \mu}, \nabla_\alpha F \right), \tag{14.6.3}$$

where $(\mu, \alpha) \in \mathfrak{se}(2)^* \cong \mathbb{R} \times \mathbb{R}^2$ and $\nabla_\alpha F$ denotes the gradient of F with respect to α. The $(\pm)$ Lie–Poisson structure on $\mathfrak{se}(2)^*$ is given by

$$\{F, G\}_\pm(\mu, \alpha) = \pm \left(\frac{\partial F}{\partial \mu} \mathbb{J}\alpha \cdot \nabla_\alpha G - \frac{\partial G}{\partial \mu} \mathbb{J}\alpha \cdot \nabla_\alpha F \right). \tag{14.6.4}$$

One also checks that functions on $\mathfrak{se}(2)^*$, of the form

$$C(\mu, \alpha) = \Phi \left(\frac{1}{2} \|\alpha\|^2 \right) \tag{14.6.5}$$

for a (smooth) function $\Phi : [0, \infty[\to \mathbb{R}$, are Casmir functions and that the symplectic leaves of $\mathfrak{se}(2)^*$ are the cylinders

$$\{(\mu, \alpha) \in \mathbb{R}^3 \mid \|\alpha\| = \text{constant} \neq 0\} \tag{14.6.6}$$

and the points on the μ-axis.

On the coadjoint orbit representing a cylinder about the μ-axis, the orbit symplectic structure is

$$\begin{aligned}
&\omega(\mu,\alpha)(\mathrm{ad}(\xi,\mathbf{u})^*(\mu,\alpha),\mathrm{ad}(\eta,\mathbf{v})^*(\mu,\alpha)) \\
&\quad = \pm(\xi\mathbb{J}\alpha\cdot\mathbf{v}-\eta\mathbb{J}\alpha\cdot\mathbf{u}) \\
&\quad = \pm(\text{area element } dA \text{ on the cylinder})/\|\alpha\|. \qquad (14.6.7)
\end{aligned}$$

The last equality is proved in the following way. Since

$$\mathrm{ad}(\xi,\mathbf{u})^*(\mu,\alpha)=(-\mathbb{J}\alpha\cdot\mathbf{u},\xi\mathbb{J}\alpha) \qquad (14.6.8)$$

and the outward unit normal to the cylinder is $(0,\alpha)/\|\alpha\|$, the area element dA is given by

$$\begin{aligned}
&dA(\mu,\alpha)((-\mathbb{J}\alpha\cdot\mathbf{u},\xi\mathbb{J}\alpha),(-\mathbb{J}\alpha\cdot\mathbf{v},\eta\mathbb{J}\alpha)) \\
&\quad = \frac{(0,\alpha)}{\|\alpha\|}\cdot[((-\mathbb{J}\alpha\cdot\mathbf{u},\xi\mathbb{J}\alpha)\times(-\mathbb{J}\alpha\cdot\mathbf{u},\xi\mathbb{J}\alpha)] \\
&\quad = \|\alpha\|(\xi\mathbb{J}\alpha\cdot\mathbf{v}-\eta\mathbb{J}\alpha\cdot\mathbf{u}).
\end{aligned}$$

The Poisson structures of $\mathfrak{so}(3)^*, \mathfrak{sl}(2,\mathbb{R})^*$, and $\mathfrak{se}(2)^*$ fit together in a larger Poisson manifold. Weinstein [1983b] considers for every $\varepsilon\in\mathbb{R}$ the Lie algebra $\mathfrak{g}_\varepsilon$ with abstract basis $\mathbf{X}_1,\mathbf{X}_2,\mathbf{X}_3$ and relations

We need to *prove by hand* that the coadjoint orbits of $\mathfrak{se}(2)^*$, which we know already are cylinders topologically, are in fact *symplectically* T^*S^1's. That is, there are no magnetic terms—but we need to do it by hand, not by reduction!

$$[\mathbf{X}_3,\mathbf{X}_1]=\mathbf{X}_2,\quad [\mathbf{X}_2,\mathbf{X}_3]=\mathbf{X}_1,\quad [\mathbf{X}_1,\mathbf{X}_2]=\varepsilon\mathbf{X}_3. \qquad (14.6.9)$$

If $\varepsilon>0$, the map

$$\mathbf{X}_1\mapsto\sqrt{\varepsilon}(1,0,0)^\wedge,\quad \mathbf{X}_2\mapsto\sqrt{\varepsilon}(0,1,0)^\wedge,\quad \mathbf{X}_3\mapsto\sqrt{\varepsilon}(0,0,1)^\wedge, \qquad (14.6.10)$$

defines an isomorphism of $\mathfrak{g}_\varepsilon$ with $\mathfrak{so}(3)$, while if $\varepsilon=0$, the map

$$\mathbf{X}_1\mapsto(0,0,-1),\quad \mathbf{X}_2\mapsto(0,-1,0),\quad \mathbf{X}_3\mapsto(-1,0,0), \qquad (14.6.11)$$

defines an isomorphism of $\mathfrak{g}_0$ with $\mathfrak{se}(2)$, and if $\varepsilon<0$, the map

$$\mathbf{X}_1\mapsto\frac{\sqrt{-\varepsilon}}{2}\begin{bmatrix}1&0\\0&-1\end{bmatrix},\quad \mathbf{X}_2\mapsto\frac{\sqrt{-\varepsilon}}{2}\begin{bmatrix}1&0\\0&1\end{bmatrix},\quad \mathbf{X}_3\mapsto\frac{1}{2}\begin{bmatrix}0&-1\\1&0\end{bmatrix}, \qquad (14.6.12)$$

defines an isomorphism of $\mathfrak{g}_\varepsilon$ with $\mathfrak{sl}(2,\mathbb{R})$.

The $(+)$ Lie–Poisson structure of $\mathfrak{g}_\varepsilon^*$ is given by the bracket relations

$$\{x_3,x_1\}=x_2,\quad \{x_2,x_3\}=x_1,\quad \{x_1,x_2\}=\varepsilon x_3, \qquad (14.6.13)$$

for the coordinate functions $x_i\in\mathfrak{g}_\varepsilon^*=\mathbb{R}^3, \langle x_i,x_j\rangle=\delta_{ij}$.

In $\mathbb{R}^4$ with coordinate functions $(x_1,x_2,x_3,\varepsilon)$ consider the above bracket relations plus $\{\varepsilon,x_1\}=\{\varepsilon,x_2\}=\{\varepsilon,x_3\}=0$. This defines a Poisson structure on $\mathbb{R}^4$ which is not of Lie–Poisson type. The leaves of this Poisson structure are all two dimensional in the space (x_1,x_2,x_3) and the Casimir functions are all functions of $x_1^2+x_2^2+\varepsilon x_3^2$ and ε. The inclusion of $\mathfrak{g}_\varepsilon^*$ in $\mathbb{R}^4$ with the above Poisson structure is a canonical map. The leaves of $\mathbb{R}^4$ with the above Poisson structure as ε passes through zero is given in Figure 14.8.1.

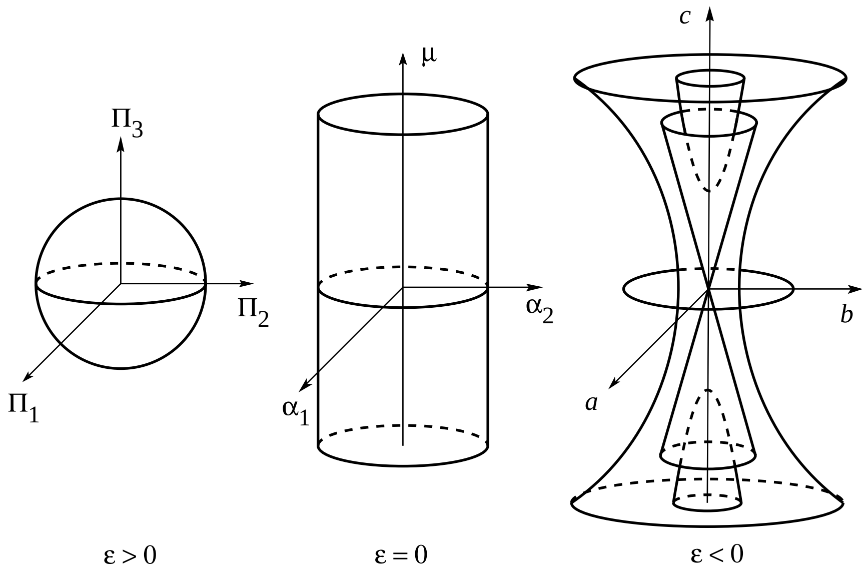

FIGURE 14.6.1. The orbit structure for $\mathfrak{so}(3)^*, \mathfrak{se}(3)^*$, and $\mathfrak{sl}(2,\mathbb{R})^*$.

14.7 The Euclidean Group of Three-Space

The Euclidean Group, its Lie Algebra and its Dual. An element of SE(3) is a pair $(\mathbf{A},\mathbf{a})$ where $\mathbf{A}\in \mathrm{SO}(3)$ and $\mathbf{a}\in\mathbb{R}^3$; the action of SE(3) on $\mathbb{R}^3$ is the rotation $\mathbf{A}$ followed by translation by the vector $\mathbf{a}$ and has the expression

$$(\mathbf{A},\mathbf{a})\cdot\mathbf{x} = \mathbf{A}\mathbf{x}+\mathbf{a}. \tag{14.7.1}$$

Using this formula, one sees that multiplication and inversion in SE(3) are given by

$$(\mathbf{A},\mathbf{a})(\mathbf{B},\mathbf{b}) = (\mathbf{A}\mathbf{B},\mathbf{A}\mathbf{b}+\mathbf{a}) \tag{14.7.2}$$

and

$$(\mathbf{A},\mathbf{a})^{-1} = (\mathbf{A}^{-1},-\mathbf{A}^{-1}\mathbf{a}), \tag{14.7.3}$$

for $\mathbf{A},\mathbf{B}\in\mathrm{SO}(3)$ and $\mathbf{a},\mathbf{b}\in\mathbb{R}^3$. The identity element is $(\mathbf{1},\mathbf{0})$. Note that SE(3) embeds into $\mathrm{SL}(4;\mathbb{R})$ via the map

$$(\mathbf{A},\mathbf{a})\mapsto \begin{bmatrix}\mathbf{A} & \mathbf{a}\\ 0 & 1\end{bmatrix}; \tag{14.7.4}$$

thus one can operate with SE(3) as one would with matrix Lie groups by using this embedding. In particular, the Lie algebra $\mathfrak{se}(3)$ of SE(3) is isomorphic to a Lie subalgebra of $\mathfrak{sl}(4;\mathbb{R})$ with elements of the form

$$\begin{bmatrix} \hat{\mathbf{x}} & \mathbf{y} \\ 0 & 0 \end{bmatrix}, \quad \text{where } \mathbf{x}, \mathbf{y} \in \mathbb{R}^3, \tag{14.7.5}$$

and a Lie algebra bracket equal to the commutator bracket of matrices. This shows that the Lie bracket operation on $\mathfrak{se}(3)$ is given by

$$[(\mathbf{x}, \mathbf{y}), (\mathbf{x}', \mathbf{y}')] = (\mathbf{x} \times \mathbf{x}', \mathbf{x} \times \mathbf{y}' - \mathbf{x}' \times \mathbf{y}). \tag{14.7.6}$$

Since

$$\begin{bmatrix} \mathbf{A} & \mathbf{a} \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{A}^{-1} & -\mathbf{A}^{-1}\mathbf{a} \\ 0 & 1 \end{bmatrix}$$

and

$$\begin{bmatrix} \mathbf{A} & \mathbf{a} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{x}} & \mathbf{y} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{A}^{-1} & -\mathbf{A}^{-1}\mathbf{a} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{A}\hat{\mathbf{x}}\mathbf{A}^{-1} & -\mathbf{A}\hat{\mathbf{x}}\mathbf{A}^{-1}\mathbf{a} + \mathbf{A}\mathbf{y} \\ 0 & 0 \end{bmatrix},$$

we see that the adjoint action of SE(3) on $\mathfrak{se}(3)$ has the expression

$$\mathrm{Ad}_{(\mathrm{A},\mathrm{a})}(\mathbf{x}, \mathbf{y}) = (\mathbf{A}\mathbf{x}, \mathbf{A}\mathbf{y} - \mathbf{A}\mathbf{x} \times \mathbf{a}). \tag{14.7.7}$$

The (6×6)-matrix of $\mathrm{Ad}_{(\mathrm{A},\mathrm{a})}$ is given by

$$\begin{bmatrix} \mathbf{A} & 0 \\ \hat{\mathbf{a}}\mathbf{A} & \mathbf{A} \end{bmatrix}. \tag{14.7.8}$$

Identifying the dual of $\mathfrak{se}(3)$ with $\mathbb{R}^3 \times \mathbb{R}^3$ by the dot product in every factor, it follows that the matrix of $\mathrm{Ad}^*_{(\mathrm{A},\mathrm{a})^{-1}}$ is given by the inverse transpose of the (6×6)-matrix (14.7.8), that is, it equals

$$\begin{bmatrix} \mathbf{A} & \hat{\mathbf{a}}\mathbf{A} \\ 0 & \mathbf{A} \end{bmatrix}. \tag{14.7.9}$$

Thus, the coadjoint action of SE(3) on $\mathfrak{se}(3)^* = \mathbb{R}^3 \times \mathbb{R}^3$ has the expression

$$\mathrm{Ad}^*_{(\mathrm{A},\mathrm{a})^{-1}}(\mathbf{u}, \mathbf{v}) = (\mathbf{A}\mathbf{u} + \mathbf{a} \times \mathbf{A}\mathbf{v}, \mathbf{A}\mathbf{v}). \tag{14.7.10}$$

(This Lie algebra is a semidirect product and all formulae derived here "by hand" are special cases of general ones that may be found in works on semidirect products; see, for example, Marsden, Ratiu, and Weinstein [1984a,b].)

Coadjoint Orbits in $\mathfrak{se}(3)^*$. Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ be an orthonormal basis of $\mathfrak{se}(3) = \mathbb{R}^3 \times \mathbb{R}^3$ such that $\mathbf{e}_i = \mathbf{f}_i, i = 1, 2, 3$. The dual basis of $\mathfrak{se}(3)^*$ via the dot product is again $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$. Let $\mathbf{e}$ and $\mathbf{f}$ denote arbitrary vectors satisfying $\mathbf{e} \in \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\mathbf{f} \in \operatorname{span}\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$. For the coadjoint action the only zero-dimensional orbit is the origin. Since $\mathfrak{se}(3)$ is six dimensional, there can also be two- and four-dimensional coadjoint orbits. These in fact occur and fall into three types.

<u>Type I</u>: The orbit through $(\mathbf{e}, \mathbf{0})$ equals

$$\mathrm{SE}(3) \cdot (\mathbf{e}, \mathbf{0}) = \{(\mathbf{A}\mathbf{e}, \mathbf{0}) \mid \mathbf{A} \in \mathrm{SO}(3)\} = S^2_{\|\mathbf{e}\|}, \tag{14.7.11}$$

the two-sphere of radius $\|\mathbf{e}\|$.

<u>Type II</u>: The orbit through $(\mathbf{0}, \mathbf{f})$

$$\begin{aligned} &\mathrm{SE}(3) \cdot (\mathbf{0}, \mathbf{f}) \\ &\qquad = \{(\mathbf{a} \times \mathbf{A}\mathbf{f}, \mathbf{A}\mathbf{f}) | \mathbf{A} \in \mathrm{SO}(3), \mathbf{a} \in \mathbb{R}^3\} \\ &\qquad = \{(\mathbf{u}, \mathbf{A}\mathbf{f}) | \mathbf{A} \in \mathrm{SO}(3), \mathbf{u} \perp \mathbf{A}\mathbf{f}\} = TS^2_{\|\mathbf{f}\|}, \end{aligned} \tag{14.7.12}$$

the tangent bundle of the two-sphere of radius $\|\mathbf{f}\|$; note the vector part is in the first slot.

<u>Type III</u>: The orbit through $(\mathbf{e}, \mathbf{f})$, where $\mathbf{e}, \mathbf{f} \neq \mathbf{0}$, equals

$$\mathrm{SE}(3) \cdot (\mathbf{e}, \mathbf{f}) = \{(\mathbf{A}\mathbf{e} + \mathbf{a} \times \mathbf{A}\mathbf{f}, \mathbf{A}\mathbf{f}) \mid \mathbf{A} \in \mathrm{SO}(3), \mathbf{a} \in \mathbb{R}^3\}. \tag{14.7.13}$$

We will prove below that this orbit is diffeomorphic to $TS^2_{\|\mathbf{f}\|}$. Consider the smooth map

$$\varphi : (\mathbf{A}, \mathbf{a}) \in \mathrm{SE}(3) \mapsto \left(\mathbf{A}\mathbf{e} + \mathbf{a} \times \mathbf{A}\mathbf{f} - \frac{\mathbf{e} \cdot \mathbf{f}}{\|\mathbf{f}\|^2} \mathbf{A}\mathbf{f}, \mathbf{A}\mathbf{f} \right) \in TS^2_{\|\mathbf{f}\|} \tag{14.7.14}$$

which is right invariant under the isotropy group

$$\mathrm{SE}(3)_{(\mathbf{e},\mathbf{f})} = \{(\mathbf{B}, \mathbf{b}) \mid \mathbf{B}\mathbf{e} + \mathbf{b} \times \mathbf{f} = \mathbf{e}, \;\; \mathbf{B}\mathbf{f} = \mathbf{f}\} \tag{14.7.15}$$

(see (14.7.10)), that is,

$$\varphi((\mathbf{A}, \mathbf{a})(\mathbf{B}, \mathbf{b})) = \varphi(\mathbf{A}, \mathbf{a})$$

for all $(\mathbf{A}, \mathbf{a}) \in \mathrm{SE}(3)$ and $(\mathbf{B}, \mathbf{b}) \in \mathrm{SE}(3)_{(\mathbf{e},\mathbf{f})}$. Thus, φ induces a smooth map $\bar{\varphi} : \mathrm{SE}(3)/\mathrm{SE}(3)_{(\mathbf{e},\mathbf{f})} \to TS^2_{\|\mathbf{f}\|}$. The map $\bar{\varphi}$ is injective, for if $\varphi(\mathbf{A}, \mathbf{a}) = \varphi(\mathbf{A}', \mathbf{a}')$, then

$$(\mathbf{A}, \mathbf{a})^{-1}(\mathbf{A}', \mathbf{a}') = (\mathbf{A}^{-1}\mathbf{A}', \mathbf{A}^{-1}(\mathbf{a}' - \mathbf{a})) \in \mathrm{SE}(3)_{(\mathbf{e},\mathbf{f})}$$

as is easily checked. To see that φ (and hence $\bar{\varphi}$) is surjective, let $(\mathbf{u}, \mathbf{v}) \in TS^2_{\|\mathrm{f}\|}$, that is, $\|\mathbf{v}\| = \|\mathbf{f}\|$ and $\mathbf{u} \cdot \mathbf{v} = 0$. Then choose an $\mathbf{A} \in \mathrm{SO}(3)$ such that $\mathbf{Af} = \mathbf{v}$ and let $\mathbf{a} = [\mathbf{v} \times (\mathbf{u} - \mathbf{Ae})]/\|\mathbf{f}\|^2$. It is then straightforward to check that $\varphi(\mathbf{A}, \mathbf{a}) = (\mathbf{u}, \mathbf{v})$ by (14.7.14). Thus, $\bar{\varphi}$ is a bijective map. Since the derivative of φ at $(\mathbf{A}, \mathbf{a})$ in the direction $T_{(\mathrm{I},0)}L_{(\mathrm{A},\mathrm{a})}(\hat{\mathbf{x}}, \mathbf{y}) = (\mathbf{A}\hat{\mathbf{x}}, \mathbf{A}\mathbf{y})$ equals

$$\begin{aligned} T_{(\mathrm{A},\mathrm{a})}\varphi(\mathbf{A}\hat{\mathbf{x}}, \mathbf{A}\mathbf{y}) &= \left.\frac{d}{dt}\right|_{t=0} \varphi(\mathbf{A}e^{t\hat{\mathbf{x}}}, \mathbf{a} + t\mathbf{A}\mathbf{y}) \\ &= (\mathbf{A}(\mathbf{x} \times \mathbf{e} + \mathbf{y} \times \mathbf{f}) + \mathbf{a} \times \mathbf{A}(\mathbf{x} \times \mathbf{f}) \\ &\qquad - \frac{\mathbf{e} \cdot \mathbf{f}}{\|\mathbf{f}\|^2}\mathbf{A}(\mathbf{x} \times \mathbf{f}), \mathbf{A}(\mathbf{x} \times \mathbf{f})) \end{aligned} \tag{14.7.16}$$

we see that its kernel consists of left translates by $(\mathbf{A}, \mathbf{a})$ of

$$\{(\mathbf{x}, \mathbf{y}) \in \mathfrak{se}(3) \mid \mathbf{x} \times \mathbf{e} + \mathbf{y} \times \mathbf{f} = \mathbf{0}, \mathbf{x} \times \mathbf{f} = 0\}. \tag{14.7.17}$$

However, taking the derivatives of the defining relations in (14.7.15) at $(\mathbf{B}, \mathbf{b}) = (\mathbf{l}, \mathbf{0})$ we see that (14.7.17) coincides with $\mathfrak{se}(3)_{(\mathrm{e},\mathrm{f})}$. This shows that $\overline{\varphi}$ is an immersion and hence, since $\dim(\mathrm{SE}(3)/\mathrm{SE}(3)_{(\mathrm{e},\mathrm{f})}) = \dim TS^2_{\|\mathrm{f}\|} = 4$, it follows that $\overline{\varphi}$ is a local diffeomorphism. Therefore, φ is a diffeomorphism.

To compute the tangent spaces to these orbits, we use Proposition 14.2.1 which states that the annihilator of the coadjoint isotropy subalgebra at μ equals $T_\mu\mathcal{O}$. The coadjoint action of the Lie algebra $\mathfrak{se}(3)$ on its dual $\mathfrak{se}(3)^*$ is computed to be

$$\mathrm{ad}^*_{(\mathrm{x},\mathrm{y})}(\mathbf{u}, \mathbf{v}) = (\mathbf{u} \times \mathbf{x} + \mathbf{v} \times \mathbf{y}, \mathbf{v} \times \mathbf{x}). \tag{14.7.18}$$

Thus, the isotropy subalgebra $\mathfrak{se}(3)_{(\mathrm{u},\mathrm{v})}$ is given again by (14.7.17), that is, it equals $\{(\mathbf{x}, \mathbf{y}) \in \mathfrak{se}(3) \mid \mathbf{u} \times \mathbf{x} + \mathbf{v} \times \mathbf{y} = \mathbf{0}, \mathbf{v} \times \mathbf{x} = \mathbf{0}\}$. Let $\mathcal{O}$ denote a nonzero coadjoint orbit in $\mathfrak{se}(3)^*$. Then the tangent space at a point in $\mathcal{O}$ is given as follows for each of the three types of orbits:

<u>Type I</u>: Since

$$\mathfrak{se}(3)_{(\mathrm{e},0)} = \{(\mathbf{x}, \mathbf{y}) \in \mathfrak{se}(3) \mid \mathbf{e} \times \mathbf{x} = \mathbf{0}\} = \mathrm{span}(\mathbf{e}) \times \mathbb{R}^3, \tag{14.7.19}$$

it follows that the tangent space to $\mathcal{O}$ at $(\mathbf{e}, \mathbf{0})$ is the tangent space to the sphere of radius $\|\mathbf{e}\|$ at the point $\mathbf{e}$ in the first factor.

<u>Type II</u>: Since

$$\begin{aligned} \mathfrak{se}(3)_{(0,\mathrm{f})} &= \{(\mathbf{x}, \mathbf{y}) \in \mathfrak{se}(3) \mid \mathbf{f} \times \mathbf{y} = \mathbf{0}, \mathbf{f} \times \mathbf{x} = \mathbf{0}\} \\ &= \mathrm{span}(\mathbf{f}) \times \mathrm{span}(\mathbf{f}), \end{aligned} \tag{14.7.20}$$

it follows that the tangent space to $\mathcal{O}$ at $(\mathbf{0}, \mathbf{f})$ equals $\mathbf{f}^\perp \times \mathbf{f}^\perp$, where $\mathbf{f}^\perp$ denotes the plane perpendicular to $\mathbf{f}$.

<u>Type III</u>: Since

$$\begin{aligned}\mathfrak{se}(3)_{(\mathrm{e},\mathrm{f})} &= \{(\mathbf{x}, \mathbf{y}) \in \mathfrak{se}(3) \mid \mathbf{e} \times \mathbf{x} + \mathbf{f} \times \mathbf{y} = \mathbf{0} \text{ and } \mathbf{f} \times \mathbf{x} = \mathbf{0}\} \\ &= \{(c_1\mathbf{f}, c_1\mathbf{e} + c_2\mathbf{f}) \mid c_1, c_2 \in \mathbb{R}\},\end{aligned} \tag{14.7.21}$$

the tangent space at $(\mathbf{e}, \mathbf{f})$ to $\mathcal{O}$ is the orthogonal complement of the space spanned by $(\mathbf{f}, \mathbf{e})$ and $(\mathbf{0}, \mathbf{f})$, that is, it equals

$$\{(\mathbf{u}, \mathbf{v}) \mid \mathbf{u} \cdot \mathbf{f} + \mathbf{v} \cdot \mathbf{e} = \mathbf{0} \quad \text{and} \quad \mathbf{v} \cdot \mathbf{f} = \mathbf{0}\}.$$

The Symplectic Form on Orbits. Let $\mathcal{O}$ denote a nonzero orbit of $\mathfrak{se}(3)^*$. We consider the different oribt types separately, as above.

<u>Type I</u>: If $\mathcal{O}$ contains a point of the form $(\mathbf{e}, \mathbf{0})$, the orbit $\mathcal{O}$ equals $S^2_{\|\mathrm{e}\|} \times \{\mathbf{0}\}$. The minus orbit symplectic form is

$$\omega^-(\mathbf{e}, \mathbf{0})(\mathrm{ad}^*_{(\mathrm{x},\mathrm{y})}(\mathbf{e}, \mathbf{0}), \mathrm{ad}^*_{(\mathrm{a},\mathrm{b})}(\mathbf{e}, \mathbf{0})) = -\mathbf{e} \cdot (\mathbf{x} \times \mathbf{x}'). \tag{14.7.22}$$

Thus, the symplectic form on $\mathcal{O}$ at $(\mathbf{e}, \mathbf{0})$ is $-1/\|\mathbf{e}\|$ times the area element of the sphere of radius $\|\mathbf{e}\|$ (see (14.3.16) and (14.3.18)).

<u>Type II</u>: If $\mathcal{O}$ contains a point of the form $(\mathbf{0}, \mathbf{f})$, then $\mathcal{O}$ equals $TS^2_{\|\mathrm{f}\|}$. Let $(\mathbf{u}, \mathbf{v}) \in \mathcal{O}$, that is, $\|\mathbf{v}\| = \|\mathbf{f}\|$ and $\mathbf{u} \perp \mathbf{v}$. The symplectic form in this case is

$$\begin{aligned}&\omega^-(\mathbf{u}, \mathbf{v})(\mathrm{ad}^*_{(\mathrm{x},\mathrm{y})}(\mathbf{u}, \mathbf{v}), \mathrm{ad}^*_{(\mathrm{a},\mathrm{b})}(\mathbf{u}, \mathbf{v})) \\ &\qquad\qquad = -\mathbf{u} \cdot (\mathbf{x} \times \mathbf{x}') - \mathbf{v} \cdot (\mathbf{x} \times \mathbf{y}' - \mathbf{x}' \times \mathbf{y}).\end{aligned} \tag{14.7.23}$$

We shall prove below that this form is exact, namely, $\omega^- = -\mathbf{d}\theta$, where

$$\theta(\mathbf{u}, \mathbf{v}) \cdot \mathrm{ad}^*_{(\mathrm{x},\mathrm{y})}(\mathbf{u}, \mathbf{v}) = \mathbf{u} \cdot \mathbf{x}. \tag{14.7.24}$$

First, note that θ is indeed well defined, for if

$$\mathrm{ad}^*_{(\mathrm{x},\mathrm{y})}(\mathbf{u}, \mathbf{v}) = \mathrm{ad}^*_{(\mathrm{x}',\mathrm{y}')}(\mathbf{u}, \mathbf{v}),$$

by (14.7.18) we have $(\mathbf{x} - \mathbf{x}') \times \mathbf{v} = 0$, that is, $\mathbf{x} - \mathbf{x}' = c\mathbf{v}$ for some constant $c \in \mathbb{R}$, and since $\mathbf{u} \perp \mathbf{v}$, we conclude from here that $\mathbf{u} \cdot \mathbf{x} = \mathbf{u} \cdot \mathbf{x}'$. Second, in order to compute $\mathbf{d}\theta$, we shall use the formula

$$\mathbf{d}\theta(X, Y) = X[\theta(Y)] - Y[\theta(X)] - \theta([X, Y])$$

for any vector fields X, Y on $\mathcal{O}$. Third, we shall choose X and Y as follows:

$$\begin{aligned}X(\mathbf{u}, \mathbf{v}) &= (\mathbf{x}, \mathbf{y})_{\mathfrak{se}(3)^*}(\mathbf{u}, \mathbf{v}) = -\,\mathrm{ad}^*_{(\mathrm{x},\mathrm{y})}(\mathbf{u}, \mathbf{v}), \\ Y(\mathbf{u}, \mathbf{v}) &= (\mathbf{x}', \mathbf{y}')_{\mathfrak{se}(3)^*}(\mathbf{u}, \mathbf{v}) = -\,\mathrm{ad}^*_{(\mathrm{x}',\mathrm{y}')}(\mathbf{u}, \mathbf{v}),\end{aligned}$$

for fixed $\mathbf{x}, \mathbf{y}, \mathbf{x}', \mathbf{y}' \in \mathbb{R}^3$. Fourth, to compute $X[\theta(Y)](\mathbf{u}, \mathbf{v})$, consider the path

$$(\mathbf{u}(\epsilon), \mathbf{v}(\epsilon)) = (e^{-\epsilon\hat{\mathbf{x}}}\mathbf{u} + \epsilon(\mathbf{v} \times \mathbf{y}), e^{-\epsilon\hat{\mathbf{x}}}\mathbf{v}),$$

which satisfies $(\mathbf{u}(0), \mathbf{v}(0)) = (\mathbf{u}, \mathbf{v})$ and

$$(\mathbf{u}'(0), \mathbf{v}'(0)) = (\mathbf{u} \times \mathbf{x} + \mathbf{v} \times \mathbf{y}, \mathbf{v} \times \mathbf{x}) = \mathrm{ad}^*_{(\mathbf{x},\mathbf{y})}(\mathbf{u}, \mathbf{v}).$$

Then

$$\begin{aligned} X[\theta(Y)](\mathbf{u}, \mathbf{v}) &= \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \theta(Y)(\mathbf{u}(\epsilon), \mathbf{v}(\epsilon)) \\ &= \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \mathbf{u}(\epsilon) \cdot \mathbf{x}' = (\mathbf{u} \times \mathbf{x} + \mathbf{v} \times \mathbf{y}) \cdot \mathbf{x}'. \end{aligned}$$

Similarly, $Y[\theta(X)](\mathbf{u}, \mathbf{v}) = (\mathbf{u} \times \mathbf{x}' + \mathbf{v} \times \mathbf{y}') \cdot \mathbf{x}$. Finally,

$$\begin{aligned} [X, Y](\mathbf{u}, \mathbf{v}) &= [(\mathbf{x}, \mathbf{y})_{\mathfrak{se}(3)^*}, (\mathbf{x}', \mathbf{y}')_{\mathfrak{se}(3)^*}](\mathbf{u}, \mathbf{v}) \\ &= -[(\mathbf{x}, \mathbf{y}), (\mathbf{x}', \mathbf{y}')]_{\mathfrak{se}(3)^*}(\mathbf{u}, \mathbf{v}) \\ &= -(\mathbf{x} \times \mathbf{x}', \mathbf{x} \times \mathbf{y}' - \mathbf{x}' \times \mathbf{y})_{\mathfrak{se}(3)^*}(\mathbf{u}, \mathbf{v}) \\ &= \mathrm{ad}^*_{(\mathbf{x}\times\mathbf{x}', \mathbf{x}\times\mathbf{y}'-\mathbf{x}'\times\mathbf{y})}(\mathbf{u}, \mathbf{v}). \end{aligned}$$

Therefore,

$$\begin{aligned} &- \mathbf{d}\theta(\mathbf{u}, \mathbf{v})(\mathrm{ad}^*_{(\mathbf{x},\mathbf{y})}(\mathbf{u}, \mathbf{v}), \mathrm{ad}^*_{(\mathbf{x}',\mathbf{y}')}(\mathbf{u}, \mathbf{v})) \\ &\quad = -X[\theta(Y)](\mathbf{u}, \mathbf{v}) + Y[\theta(X)](\mathbf{u}, \mathbf{v}) + \theta([X, Y])(\mathbf{u}, \mathbf{v}) \\ &\quad = -(\mathbf{u} \times \mathbf{x} + \mathbf{v} \times \mathbf{y}) \cdot \mathbf{x}' + (\mathbf{u} \times \mathbf{x}' + \mathbf{v} \times \mathbf{y}') \cdot \mathbf{x} + \mathbf{u} \cdot (\mathbf{x} \times \mathbf{x}') \\ &\quad = -\mathbf{u} \cdot (\mathbf{x} \times \mathbf{x}') - \mathbf{v} \cdot (\mathbf{x} \times \mathbf{y}' - \mathbf{x}' \times \mathbf{y}), \end{aligned}$$

which coincides with (14.7.23).

The form θ given by (14.7.24) is the canonical symplectic structure when we identify $TS^2_{\|\mathbf{f}\|}$ with $T^*S^2_{\|\mathbf{f}\|}$ using the Euclidean metric.

<u>Type III</u>: If $\mathcal{O}$ contains $(\mathbf{e}, \mathbf{f})$, where $\mathbf{e} \neq \mathbf{0}$ and $\mathbf{f} \neq \mathbf{0}$, then $\mathcal{O}$ is diffeomorphic to $T^*S^2_{\|\mathbf{f}\|}$ in the following way. The map $\varphi : \mathrm{SE}(3) \to T^*S^2_{\|\mathbf{f}\|}$ given by (14.7.14) induces a diffeomorphism

$$\overline{\varphi} : \mathrm{SE}(3)/\,\mathrm{SE}(3)_{(\mathbf{e},\mathbf{f})} \to T^*S^2_{\|\mathbf{f}\|}.$$

However, the orbit $\mathcal{O}$ through $(\mathbf{e}, \mathbf{f})$ is diffeomorphic to $\mathrm{SE}(3)/\,\mathrm{SE}(3)_{(\mathbf{e},\mathbf{f})}$ by the diffeomorphism

$$(\mathbf{A}, \mathbf{a}) \mapsto \mathrm{Ad}^*_{(\mathbf{A},\mathbf{a})^{-1}}(\mathbf{e}, \mathbf{f}). \tag{14.7.25}$$

Therefore, the diffeomorphism $\Phi : \mathcal{O} \to T^*S^2_{\|\mathrm{f}\|}$ is given by

$$\begin{aligned}\Phi(\mathrm{Ad}^*_{(\mathrm{A},\mathrm{a})^{-1}}(\mathbf{e},\mathbf{f})) &= \Phi(\mathbf{Ae} + \mathbf{a} \times \mathbf{Af}, \mathbf{Af}) \\ &= (\mathbf{Ae} + \mathbf{a} \times \mathbf{Af} - \frac{\mathbf{e}\cdot\mathbf{f}}{\|\mathbf{f}\|^2}\mathbf{Af}, \mathbf{Af}).\end{aligned} \tag{14.7.26}$$

If $(\overline{\mathbf{u}}, \overline{\mathbf{v}}) \in \mathcal{O}$, the orbit symplectic structure is given by formula (14.7.23), where $\overline{\mathbf{u}} = \mathbf{Ae} + \mathbf{a} \times \mathbf{Af}, \overline{\mathbf{v}} = \mathbf{Af}$ for some $\mathbf{A} \in \mathrm{SO}(3), \mathbf{a} \in \mathbb{R}^3$. Let

$$\begin{aligned}\mathbf{u} &= \mathbf{Ae} + \mathbf{a} \times \mathbf{Af} - \frac{\mathbf{e}\cdot\mathbf{f}}{\|\mathbf{f}\|^2}\mathbf{Af} = \overline{\mathbf{u}} - \frac{\mathbf{e}\cdot\mathbf{f}}{\|\mathbf{f}\|^2}\overline{\mathbf{v}}, \\ \mathbf{v} &= \mathbf{Af} = \overline{\mathbf{v}},\end{aligned} \tag{14.7.27}$$

the pair of vectors $(\mathbf{u}, \mathbf{v})$ representing an element of TS^2. Note that $\|\mathbf{v}\| = \|\mathbf{f}\|$ and $\mathbf{u}\cdot\mathbf{v} = 0$. Then a tangent vector to $TS^2_{\|\mathrm{f}\|}$ at $(\mathbf{u}, \mathbf{v})$ can be represented as $\mathrm{ad}^*_{(\mathrm{x},\mathrm{y})}(\mathbf{u}, \mathbf{v}) = (\mathbf{u} \times \mathbf{x} + \mathbf{v} \times \mathbf{y}, \mathbf{v} \times \mathbf{x})$ so that by (14.7.26) we get

$$\begin{aligned}T_{(\mathrm{u},\mathrm{v})}\Phi^{-1}(\mathrm{ad}^*_{(\mathrm{x},\mathrm{y})}(\mathbf{u}, \mathbf{v})) &= \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \Phi^{-1}(e^{-\epsilon\hat{\mathrm{x}}}\mathbf{u} + \epsilon(\mathbf{v} \times \mathbf{y}), e^{\epsilon\hat{\mathrm{x}}}\mathbf{v}) \\ &= \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \left(e^{-\epsilon\hat{\mathrm{x}}}\mathbf{u} + \epsilon(\mathbf{v} \times \mathbf{y}) + \frac{\mathbf{e}\cdot\mathbf{f}}{\|f\|^2}e^{-\epsilon\hat{\mathrm{x}}}\mathbf{v}, e^{-\epsilon\hat{\mathrm{x}}}\mathbf{v}\right) \\ &= \left(\mathbf{u} \times \mathbf{x} + \mathbf{v} \times \mathbf{y} + \frac{\mathbf{e}\cdot\mathbf{f}}{\|f\|^2}(\mathbf{v} \times \mathbf{x}), \mathbf{v} \times \mathbf{x}\right) \\ &= (\overline{\mathbf{u}} \times \mathbf{x} + \overline{\mathbf{v}} \times \mathbf{y}, \overline{\mathbf{v}} \times \mathbf{x}) \\ &= \mathrm{ad}^*_{(\mathrm{x},\mathrm{y})}(\overline{\mathbf{u}}, \overline{\mathbf{v}}).\end{aligned}$$

Therefore, the push-forward of the orbit symplectic form ω^- to $TS^2_{\|\mathrm{f}\|}$ is

$$\begin{aligned}&(\Phi_*\omega^-)(\mathbf{u}, \mathbf{v})(\mathrm{ad}^*_{(\mathrm{x},\mathrm{y})}(\mathbf{u}, \mathbf{v}), \mathrm{ad}^*_{(\mathrm{x}',\mathrm{y}')}(\mathbf{u}, \mathbf{v})) \\ &\quad= \omega^-(\overline{\mathbf{u}}, \overline{\mathbf{v}})(T_{(\mathrm{u},\mathrm{v})}\Phi^{-1}(\mathrm{ad}^*_{(\mathrm{x},\mathrm{y})}(\mathbf{u}, \mathbf{v})), T_{(\mathrm{u},\mathrm{v})}\Phi^{-1}(\mathrm{ad}^*_{(\mathrm{x}',\mathrm{y}')}(\mathbf{u}, \mathbf{v})) \\ &\quad= \omega^-(\overline{\mathbf{u}}, \overline{\mathbf{v}})(\mathrm{ad}^*_{(\mathrm{x},\mathrm{y})}(\overline{\mathbf{u}}, \overline{\mathbf{v}}), \mathrm{ad}^*_{(\mathrm{x}',\mathrm{y}')}(\overline{\mathbf{u}}, \overline{\mathbf{v}})) \\ &\quad= -\overline{\mathbf{u}}\cdot(\mathbf{x} \times \mathbf{x}') - \overline{\mathbf{v}}\cdot(\mathbf{x} \times \mathbf{y}' - \mathbf{x}' \times \mathbf{y}) \\ &\quad= -\mathbf{u}\cdot(\mathbf{x} \times \mathbf{x}') - \mathbf{v}\cdot(\mathbf{x} \times \mathbf{y}' - \mathbf{x}' \times \mathbf{y}) - \frac{\mathbf{e}\cdot\mathbf{f}}{\|\mathbf{f}\|^2}\mathbf{v}\cdot(\mathbf{x} \times \mathbf{x}').\end{aligned} \tag{14.7.28}$$

The first two terms represent the canonical symplectic structure on $TS^2_{\|\mathrm{f}\|}$ (identified via the Euclidean metric with $T^*S^2_{\|\mathrm{f}\|}$), as we have seen in the analysis of type II orbits. The third term is the following two-form on $TS^2_{\|\mathrm{f}\|}$

$$\beta(\mathbf{u}, \mathbf{v})\left(\mathrm{ad}^*_{(\mathrm{x},\mathrm{y})}(\mathbf{u}, \mathbf{v}), \mathrm{ad}^*_{(\mathrm{x}',\mathrm{y}')}(\mathbf{u}, \mathbf{v})\right) = -\frac{\mathbf{e}\cdot\mathbf{f}}{\|\mathbf{f}\|^2}\mathbf{v}\cdot(\mathbf{x} \times \mathbf{x}'). \tag{14.7.29}$$

As in the case of θ for type II orbits, it is easily seen that (14.7.28) correctly defines a two-form on $TS^2_{\|\mathbf{f}\|}$. It is necessarily closed since it is the difference between $\Phi_* \omega^-$ and the canonical two-form on $TS^2_{\|\mathbf{f}\|}$. The two-form β is a magnetic term in the sense of §6.6.

We remark that the semidirect product theory of Marsden, Ratiu, and Weinstein [1984a,b], combined with cotangent bundle reduction theory, (see, for example, Marsden [1992]) can be used to give an alternative approach to the computation of the orbit symplectic forms.

Exercises

◇ **Exercise 14.7-1.** Let K be a quadratic form on $\mathbb{R}^3$ and let $\mathbf{K}$ be the associated symmetric (3×3)-matrix. Let

$$\{F, L\}_K = -\nabla K \cdot (\nabla F \times \nabla L).$$

Show that this is the Lie–Poisson bracket for the Lie algebra structure

$$[\mathbf{u}, \mathbf{v}]_K = \mathbf{K}(\mathbf{u} \times \mathbf{v}).$$

What is the underlying Lie group?

◇ **Exercise 14.7-2.** Determine the coadjoint orbits for the Lie algebra in the preceding exercise and calculate the orbit symplectic structure. Specialize to the case $\mathrm{SO}(2,1)$.

◇ **Exercise 14.7-3.** Classify the coadjoint orbits of $\mathrm{SU}(1,1)$, namely, the group of complex (2×2) matrices of determinant one, of the form

$$g = \begin{pmatrix} a & b \\ \bar{a} & \bar{b} \end{pmatrix}.$$

where $|a|^2 - |b|^2 = 1$.