12

Computation and Properties of Momentum Maps

The previous chapter gave the general theory of momentum maps. In this chapter, we develop techniques for computing them. One of the most important cases is when there is a group action on a cotangent bundle and this action is obtained from lifting an action on the base. These transformations are called *extended point transformations*.

12.1 Momentum Maps on Cotangent Bundles

Momentum Functions. We begin by defining functions on cotangent bundles associated to vector fields on the base.

Given a manifold Q, Define the map $\mathcal{P} : \mathfrak{X}(Q) \to \mathcal{F}(T^*Q)$, by

$$\mathcal{P}(X)(\alpha_q) = \langle \alpha_q, X(q) \rangle ,$$

for $q \in Q$ and $\alpha_q \in T_q^*Q$, where $\langle\, ,\rangle$ denotes the pairing between covectors $\alpha \in T_q^*Q$ and vectors. We call $\mathcal{P}(X)$ the ***momentum function of*** X.

Definition 12.1.1. *Given a manifold Q, let $\mathcal{L}(T^*Q)$ denote the space of smooth functions $F : T^*Q \to \mathbb{R}$ that are linear on fibers of T^*Q.*

Using coordinates and working in finite dimensions, we can write $F, H \in \mathcal{L}(T^*Q)$ as

$$F(q,p) = \sum_{i=1}^{n} X^i(q)p_i, \quad \text{and} \quad H(q,p) = \sum_{i=1}^{n} Y^i(q)p_i,$$

for functions X^i and Y^i. We claim that the standard Poisson bracket $\{F, H\}$ is again linear on the fibers. Indeed, using summations on repeated indices,

$$\{F, H\}(q, p) = \frac{\partial F}{\partial q^j}\frac{\partial H}{\partial p_j} - \frac{\partial H}{\partial q^j}\frac{\partial F}{\partial p_j} = \frac{\partial X^i}{\partial q^j} p_i Y^k \delta^j_k - \frac{\partial Y^i}{\partial q^j} p_i X^k \delta^j_k$$

and so

$$\{F, H\} = \left(\frac{\partial X^i}{\partial q^j} Y^j - \frac{\partial Y^i}{\partial q^j} X^j \right) p_i. \tag{12.1.1}$$

Hence $\mathcal{L}(T^*Q)$ is a Lie subalgebra of $\mathcal{F}(T^*Q)$. If Q is infinite dimensional, a similar proof, using canonical cotangent bundle charts, works.

Lemma 12.1.2 (Momentum Commutator Lemma). *The Lie algebras:*

(i) $(\mathfrak{X}(Q), [\,,])$ *of vector fields on Q; and*

(ii) *Hamiltonian vector fields X_F on T^*Q with $F \in \mathcal{L}(T^*Q)$*

are isomorphic. Moreover, each of these algebras is anti-isomorphic to $(\mathcal{L}(T^*Q), \{\,,\})$. In particular, we have

$$\{\mathcal{P}(X), \mathcal{P}(Y)\} = -\mathcal{P}([X, Y]). \tag{12.1.2}$$

Proof. Since $\mathcal{P}(X) : T^*Q \to \mathbb{R}$ is linear on fibers, it follows that $\mathcal{P} : \mathfrak{X}(Q) \to \mathcal{L}(T^*Q)$. This map is linear and satisfies (12.1.2) since

$$[X, Y]^i = (\partial Y^i/\partial q^j)\, X^j - (\partial X^i/\partial q^j)\, Y^j$$

implies that

$$-\mathcal{P}([X, Y]) = \left(\frac{\partial X^i}{\partial q^j} Y^j - \frac{\partial Y^i}{\partial q^j} X^j \right) p_i,$$

which coincides with $\{\mathcal{P}(X), \mathcal{P}(Y)\}$ by (12.1.1). (We leave it to the reader to write out the infinite-dimensional proof.) Furthermore, $\mathcal{P}(X) = 0$ implies that $X = 0$ by the Hahn–Banach theorem. Finally, (assuming our model space is reflexive) for each $F \in \mathcal{L}(T^*Q)$, define $X(F) \in \mathfrak{X}(Q)$ by $\langle \alpha_q, X(F)(q) \rangle = F(\alpha_q)$, where $\alpha_q \in T^*_q Q$. Then $\mathcal{P}(X(F)) = F$, so $\mathcal{P}$ is also surjective, thus proving that $(\mathfrak{X}(Q), [\,,])$ and $(\mathcal{L}(T^*Q), \{\,,\})$ are anti-isomorphic Lie algebras.

The map $F \mapsto X_F$ is a Lie algebra antihomomorphism from the algebra $(\mathcal{L}(T^*Q), \{\,,\})$ to $(\{X_F \mid F \in \mathcal{L}(T^*Q)\}, [\,,])$ by (5.5.6). This map is surjective by definition. Moreover, if $X_F = 0$, then F is constant on T^*Q, hence equal to zero since it is linear on the fibers. ■

In quantum mechanics, the ***Dirac rule*** associates the differential operator

$$\mathrm{X} = \frac{\hbar}{i} X^j \frac{\partial}{\partial q^j} \tag{12.1.3}$$

with the momentum function $\mathcal{P}(X)$. (Dirac [1930], §21 and §22.) Thus, if we define $P_{\mathrm{X}} = \mathcal{P}(X)$, (12.1.2) gives

$$i\hbar\{P_{\mathrm{X}}, P_{\mathrm{Y}}\} = i\hbar\{\mathcal{P}(X), \mathcal{P}(Y)\} = -i\hbar\mathcal{P}([X,Y]) = P_{[\mathrm{X},\mathrm{Y}]}. \tag{12.1.4}$$

One can augment (12.1.4) by including lifts of functions on Q. Given $f \in \mathcal{F}(Q)$, let $f^* = f \circ \pi_Q$ where $\pi_Q : T^*Q \to Q$ is the projection, so f^* is constant on fibers. One finds that

$$\{f^*, g^*\} = 0 \tag{12.1.5}$$

and

$$\{f^*, \mathcal{P}(X)\} = X[f]. \tag{12.1.6}$$

Hamiltonian Flows of Momentum Functions. The Hamiltonian flow φ_t of X_{f^*} is fiber translation by $-t\,\mathbf{d}f$, that is, $(q,p) \mapsto (q, p - t\mathbf{d}f(q))$. The flow of $X_{\mathcal{P}(X)}$ is given by the following:

Proposition 12.1.3. *If $X \in \mathfrak{X}(Q)$ has flow φ_t, then the flow of $X_{\mathcal{P}(X)}$ on T^*Q is $T^*\varphi_{-t}$.*

Proof. If $\pi_Q : T^*Q \to Q$ denotes the canonical projection, differentiating the relation

$$\pi_Q \circ T^*\varphi_{-t} = \varphi_t \circ \pi_Q \tag{12.1.7}$$

at $t = 0$ gives

$$T\pi_Q \circ Y = X \circ \pi_Q, \tag{12.1.8}$$

where

$$Y(\alpha_q) = \frac{d}{dt} T^*\varphi_{-t}(\alpha_q)\bigg|_{t=0}, \tag{12.1.9}$$

so $T^*\varphi_{-t}$ is the flow of Y. Since $T^*\varphi_{-t}$ preserves the canonical one-form Θ on T^*Q, it follows that $\pounds_Y\Theta = 0$. Hence

$$\mathbf{i}_Y\Omega = -\mathbf{i}_Y\mathbf{d}\Theta = \mathbf{d}\mathbf{i}_Y\Theta. \tag{12.1.10}$$

By definition of the canonical one-form,

$$\begin{aligned} \mathbf{i}_Y\Theta(\alpha_q) &= \langle \Theta(\alpha_q), Y(\alpha_q)\rangle = \langle \alpha_q, T\pi_Q(Y(\alpha_q))\rangle \\ &= \langle \alpha_q, X(q)\rangle = \mathcal{P}(X)(\alpha_q), \end{aligned} \tag{12.1.11}$$

that is, $\mathbf{i}_Y\Omega = \mathbf{d}\mathcal{P}(X)$ so that $Y = X_{\mathcal{P}(X)}$. ■

Because of this proposition, the Hamiltonian vector field $X_{\mathcal{P}(X)}$ on T^*Q is called the ***cotangent lift*** of $X \in \mathfrak{X}(Q)$ to T^*Q. We also use the notation $X' := X_{\mathcal{P}(X)}$ for the cotangent lift of X. From $X_{\{F,H\}} = -[X_F, X_H]$ and (12.1.2), we get

$$\begin{aligned}[X', Y'] &= [X_{\mathcal{P}(X)}, X_{\mathcal{P}(Y)}] = -X_{\{\mathcal{P}(X),\, \mathcal{P}(Y)\}} \\ &= -X_{-\mathcal{P}[X,Y]} = [X, Y]'. \end{aligned} \tag{12.1.12}$$

For finite-dimensional Q, in local coordinates, we have

$$\begin{aligned} X' := X_{\mathcal{P}(X)} &= \sum_{i=1}^{n} \left(\frac{\partial \mathcal{P}(X)}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial \mathcal{P}(X)}{\partial q^i} \frac{\partial}{\partial p_i} \right) \\ &= X^i \frac{\partial}{\partial q^i} - \frac{\partial X^i}{\partial q^j} p_i \frac{\partial}{\partial p_j}. \end{aligned} \tag{12.1.13}$$

Cotangent Momentum Maps. Perhaps the most important result for the computation of momentum maps is the following.

Theorem 12.1.4 (Momentum Maps for Lifted Actions). *Suppose that the Lie algebra $\mathfrak{g}$ acts on the left on the manifold Q, so that $\mathfrak{g}$ acts on $P = T^*Q$ on the left by the canonical action $\xi_P = \xi_Q'$, where ξ_Q' is the cotangent lift of ξ_Q to P and $\xi \in \mathfrak{g}$. This $\mathfrak{g}$-action on P is Hamiltonian with infinitessimally equivariant momentum map $\mathbf{J} : P \to \mathfrak{g}^*$ given by*

$$\langle \mathbf{J}(\alpha_q), \xi \rangle = \langle \alpha_q, \xi_Q(q) \rangle = \mathcal{P}(\xi_Q)(\alpha_q). \tag{12.1.14}$$

If $\mathfrak{g}$ is the Lie algebra of a Lie group G which acts on Q and hence on T^*Q by cotangent lift, then $\mathbf{J}$ is equivariant.

In coordinates q^i, p_j on T^*Q and ξ^a on $\mathfrak{g}$, (12.1.14) reads

$$J_a \xi^a = p_i \xi_Q^i = p_i A^i{}_a \xi^a,$$

where $\xi_Q^i = \xi^a A^i{}_a$ are the components of ξ_Q; thus,

$$J_a(q, p) = p_i A^i{}_a(q). \tag{12.1.15}$$

Proof. For $\xi, \eta \in \mathfrak{g}$, (12.1.12) gives

$$[\xi, \eta]_P = [\xi, \eta]_Q' = -[\xi_Q, \eta_Q]' = -[\xi_Q', \eta_Q'] = -[\xi_P, \eta_P]$$

and hence $\xi \mapsto \xi_P$ is a left algebra action. This action is also canonical, for if $F, H \in \mathcal{F}(P)$,

$$\begin{aligned} \xi_P[\{F, H\}] &= X_{\mathcal{P}(\xi_Q)}[\{F, H\}] \\ &= \left\{ X_{\mathcal{P}(\xi_Q)}[F], H \right\} + \left\{ F, X_{\mathcal{P}(\xi_Q)}[H] \right\} \\ &= \{\xi_P[F], H\} + \{F, \xi_P[H]\} \end{aligned}$$

by the Jacobi identity for the Poisson bracket. If φ_t is the flow of ξ_Q, the flow of $\xi'_Q = X_{\mathcal{P}(\xi_Q)}$ is $T^*\varphi_{-t}$. Consequently, $\xi_P = X_{\mathcal{P}(\xi_Q)}$ and, thus, the $\mathfrak{g}$-action on P admits a momentum map given by $J(\xi) = P(\xi_Q)$. Since $\xi \in \mathfrak{g} \mapsto \mathcal{P}(\xi_Q) = J(\xi) \in \mathcal{F}(P)$ is a Lie algebra homomorphism by (11.1.5) and (12.1.12), it follows that $\mathbf{J}$ is an infinitessimally equivariant momentum map (Theorem 11.6.1).

Equivariance under G is proved directly in the following way. For any $g \in G$, we have

$$\begin{aligned}
\langle \mathbf{J}(g \cdot \alpha_q), \xi \rangle &= \langle g \cdot \alpha_q, \xi_Q(g \cdot q) \rangle \\
&= \left\langle \alpha_q, (T_{g\cdot q}\Phi_g^{-1} \circ \xi_Q \circ \Phi_g)(q) \right\rangle \\
&= \left\langle \alpha_q, (\Phi_g^* \xi_Q)(q) \right\rangle \\
&= \left\langle \alpha_q, (\mathrm{Ad}_{g^{-1}} \xi)_Q(q) \right\rangle \quad \text{(by Lemma 9.3.7ii)} \\
&= \left\langle \mathbf{J}(\alpha_q), \mathrm{Ad}_{g^{-1}} \xi \right\rangle \\
&= \left\langle \mathrm{Ad}^*_{g^{-1}}(\mathbf{J}(\alpha_q), \xi \right\rangle .
\end{aligned}$$

■

Remarks.

1. Let $G = \mathrm{Diff}(Q)$ act on T^*Q by cotangent lift. Then the infinitesimal generator of $X \in \mathfrak{X}(Q) = \mathfrak{g}$ is $X_{\mathcal{P}(X)}$ by Proposition 12.1.3 so that the associated momentum map is $\mathbf{J} : T^*Q \to \mathfrak{X}(Q)^*$ which is defined through $J(X) = \mathcal{P}(X)$ by the above calculations.

2. Momentum Fiber Translations. Let $G = \mathcal{F}(Q)$ act on T^*Q by fiber translations by $\mathbf{d}f$, that is,

$$f \cdot \alpha_q = \alpha_q + \mathbf{d}f(q). \tag{12.1.16}$$

Since the infinitessimal generator of $\xi \in \mathcal{F}(Q) = \mathfrak{g}$ is the vertical lift of $\mathbf{d}\xi(q)$ at α_q and this in turn equals the Hamiltonian vector field $-X_{\xi\circ\pi_Q}$, we see that the momentum map $\mathbf{J} : T^*Q \to \mathcal{F}(Q)^*$ is given by

$$J(\xi) = -\xi \circ \pi_Q. \tag{12.1.17}$$

This momentum map is equivariant since π_Q is constant on fiber translations.

3. The commutation relations

$$\begin{array}{rcl}
\{\mathcal{P}(X), \mathcal{P}(Y)\} &=& -\mathcal{P}([X,Y]), \\
\{\mathcal{P}(X),\, \xi \circ \pi_Q\} &=& -X[\xi] \circ \pi_Q, \\
\{\xi \circ \pi_Q,\, \eta \circ \pi_Q\} &=& 0,
\end{array} \tag{12.1.18}$$

can be rephrased as saying that the pair $(\mathbf{J}(X), \mathbf{J}(f))$ fit together to form a momentum map for the semidirect product group

$$\mathrm{Diff}(Q) \circledS \mathcal{F}(Q).$$

This plays an important role in the general theory of semidirect products for which we refer the reader to Marsden, Weinstein, Ratiu, Schmid and Spencer [1983], and Marsden, Ratiu and Weinstein [1984a, b]. ♦

The terminology ***extended point transformations*** arises for the following reasons. Let $\Phi : G \times Q \to Q$ be a smooth action and consider its lift $\Phi : G \times T^*Q \to T^*Q$ to the cotangent bundle. The action Φ moves points in the configuration space Q, and Φ is its natural extension to phase space T^*Q; in coordinates, the action on configuration points $q^i \mapsto \overline{q}^i$ induces the following action on momenta:

$$p_i \mapsto \overline{p}_i = \frac{\partial \overline{q}^j}{\partial q^i} p_j. \qquad (12.1.19)$$

Exercises

◇ **Exercise 12.1-1.** What is the analogue of (12.1.18):

$$\begin{aligned} \{\mathcal{P}(X), \mathcal{P}(Y)\} &= -\mathcal{P}([X, Y]), \\ \{\mathcal{P}(X),\, \xi \circ \pi_Q\} &= -X[\xi] \circ \pi_Q, \\ \{\xi \circ \pi_Q,\, \eta \circ \pi_Q\} &= 0, \end{aligned}$$

for rotations and translations on $\mathbb{R}^3$?

◇ **Exercise 12.1-2.** Prove (12.1.2):

$$\{\mathcal{P}(X), \mathcal{P}(Y)\} = -\mathcal{P}([X, Y]).$$

in infinite dimensions.

◇ **Exercise 12.1-3.** Prove Theorem 12.1.4 as a consequence of formula 11.3.4), namely,

$$\langle \mathbf{J}(z), \xi \rangle = \left(\mathbf{i}_{\xi_P} \Theta\right)(z),$$

and Exercise 11.6-3.

12.2 Momentum Maps on Tangent Bundles

Proposition 12.2.1. *Let the Lie algebra $\mathfrak{g}$ act on the left on the manifold Q and assume that $L : TQ \to \mathbb{R}$ is a regular Lagrangian. Endow TQ*

with the symplectic form $\Omega_L = (\mathbb{F}L)^*\Omega$, *where* $\Omega = -\mathbf{d}\Theta$ *is the canonical symplectic form on* T^*Q. *Then* $\mathfrak{g}$ *acts canonically on* $P = TQ$ *by*

$$\xi_P(v_q) = \left.\frac{d}{dt}\right|_{t=0} T_q\varphi_t(v_q),$$

where φ_t *is the flow of* ξ_Q *and has the infinitessimally equivariant momentum map* $\mathbf{J} : TQ \to \mathfrak{g}^*$ *given by*

$$\langle \mathbf{J}(v_q), \xi\rangle = \langle \mathbb{F}L(v_q), \xi_Q(q)\rangle . \tag{12.2.1}$$

If $\mathfrak{g}$ *is the Lie algebra of a Lie group* G *and* G *acts on* Q *and hence on* TQ *by tangent lift, then* $\mathbf{J}$ *is equivariant.*

Proof. Use (11.3.4), a direct calculation or, if L is hyperregular, the following argument. Since $\mathbb{F}L$ is a symplectic diffeomorphism, $\xi \mapsto \xi_P = (\mathbb{F}L)^*\xi_{T^*Q}$ is a canonical left Lie algebra action. Therefore, the composition of $\mathbb{F}L$ with the momentum map (12.1.14) is the momentum map of the $\mathfrak{g}$-action on TQ. ■

In coordinates $(q^i, \dot{q}^i)$ on TQ and (ξ^a) on $\mathfrak{g}$, (12.2.1) reads

$$J_a(q^i, \dot{q}^i) = \frac{\partial L}{\partial \dot{q}^i} A^i{}_a(q), \tag{12.2.2}$$

where $\xi^i_Q(q) = \xi^a A^i{}_a(q)$ are the components of ξ_Q.

Examples

(a) The Hamiltonian. A Hamiltonian $H : P \to \mathbb{R}$ on a Poisson manifold P having a complete vector field X_H is an equivariant momentum map for the $\mathbb{R}$-action given by the flow of X_H. ♦

(b) Linear Momentum. In the notations of Example (b) of §11.5 we recompute the linear momentum of the N-particle system. Since $\mathbb{R}^3$ acts on points $(\mathbf{q}_1, \ldots, \mathbf{q}_N)$ in $\mathbb{R}^{3N}$ by $\mathbf{x} \cdot (\mathbf{q}_j) = (\mathbf{q}_j + \mathbf{x})$, the infinitessimal generator is

$$\xi_{\mathbb{R}^{3N}}(\mathbf{q}_j) = (\mathbf{q}_1, \ldots, \mathbf{q}_N, \xi, \ldots, \xi) \tag{12.2.3}$$

(this has the base point $(\mathbf{q}_1, \ldots, \mathbf{q}_N)$ and vector part $(\xi, \ldots, \xi)$ (N times)). Consequently, by (12.1.14), an equivariant momentum map $\mathbf{J} : T^*\mathbb{R}^{3N} \to \mathbb{R}^3$ is given by

$$J(\xi)(\mathbf{q}_j, \mathbf{p}^j) = \sum_{j=1}^{N} \mathbf{p}^j \cdot \xi, \quad \text{i.e.,} \quad \mathbf{J}(\mathbf{q}_j, \mathbf{p}^j) = \sum_{j=1}^{N} \mathbf{p}^j. \qquad ♦$$

(c) Angular Momentum. In the notation of Example (c) of §11.5, let SO(3) act on $\mathbb{R}^3$ by matrix multiplication $\mathbf{A} \cdot \mathbf{q} = \mathbf{A}\mathbf{q}$. The infinitessimal generator is given by $\hat{\omega}_{\mathbb{R}^3}(\mathbf{q}) = \hat{\omega}\mathbf{q} = \omega \times \mathbf{q}$ where $\omega \in \mathbb{R}^3$. Consequently, by (12.1.14), an equivariant momentum map $\mathbf{J} : T^*\mathbb{R}^3 \to \mathfrak{so}(3)^* \cong \mathbb{R}^3$ is given by

$$\langle \mathbf{J}(\mathbf{q}, \mathbf{p}), \omega \rangle = \mathbf{p} \cdot \hat{\omega}\mathbf{q} = \omega \cdot (\mathbf{q} \times \mathbf{p}),$$

that is,

$$\mathbf{J}(\mathbf{q}, \mathbf{p}) = \mathbf{q} \times \mathbf{p}. \tag{12.2.4}$$

Equivariance in this case reduces to the relation $\mathbf{A}\mathbf{q} \times \mathbf{A}\mathbf{p} = \mathbf{A}(\mathbf{q} \times \mathbf{p})$ for any $A \in \mathrm{SO}(3)$. If $A \in \mathrm{O}(3) \setminus \mathrm{SO}(3)$, such as a reflection, this relation is no longer satisfied; a minus sign appears on the right-hand side, a fact sometimes phrased by stating that *angular momentum is a pseudo-vector*. On the other hand, letting O(3) act on $\mathbb{R}^3$ by matrix multiplication, $\mathbf{J}$ is given by the same formula and so is the momentum map of a lifted action and these are *always* equivariant. We have an apparent contradiction—What is wrong? The answer is that the adjoint action and the isomorphism $\hat{}: \mathbb{R}^3 \to \mathfrak{so}(3)$ are related for the component of $-$(Identity) in O(3) by $\mathbf{A}\hat{x}\mathbf{A}^{-1} = -(\mathbf{A}x)\hat{}$. Thus, $\mathbf{J}(\mathbf{q}, \mathbf{p})$ is indeed equivariant as it stands. (One does not need a separate terminology like "pseudo-vector" to see what is going on.) ♦

(d) Momentum for Matrix Groups. In the notations of Example (d) of §11.5, let the Lie group $G \subset \mathrm{GL}(n, \mathbb{R})$ act on $\mathbb{R}^n$ by $\mathbf{A} \cdot \mathbf{q} = \mathbf{A}\mathbf{q}$. The infinitessimal generator of this action is given by

$$\xi_{\mathbb{R}^N}(\mathbf{q}) = \xi\mathbf{q},$$

for $\xi \in \mathfrak{g}$, the Lie algebra of G, regarded as a subalgebra $\mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{R})$. By (12.1.14), the lift of the G-action on $\mathbb{R}^3$ to $T^*\mathbb{R}^n$ has an equivariant momentum map $\mathbf{J} : T^*\mathbb{R}^n \to \mathfrak{g}^*$ given by

$$\langle \mathbf{J}(\mathbf{q}, \mathbf{p}), \xi \rangle = \mathbf{p} \cdot (\xi\mathbf{q}) \tag{12.2.5}$$

which coincides with (11.4.14). ♦

(e) The Dual of a Lie Algebra Homomorphism. From Example (f) of §11.5 it follows that the dual of a Lie algebra homomorphism $\alpha : \mathfrak{h} \to \mathfrak{g}$ is an equivariant momentum map which does not arise from an action which is an extended point transformation. Recall that a *linear map* $\alpha : \mathfrak{h} \to \mathfrak{g}$ *is a Lie algebra homomorphism if and only if the dual map* $\alpha^* : \mathfrak{g}^* \to \mathfrak{h}^*$ *is Poisson.* ♦

(f) Momentum Maps Induced by Subgroups. Suppose that the Poisson Lie group action of G on P admits an equivariant momentum map $\mathbf{J}$, and if H is a Lie subgroup of G, then in the notation of Exercise 11.5-1,

change 'if' to 'suppose' to avoid tryng to break 'group'??

$i^* \circ \mathbf{J} : P \to \mathfrak{h}^*$ is an equivariant momentum map of the induced H-action on P. ♦

(g) Products. Let P_1 and P_2 be Poisson manifolds and let $P_1 \times P_2$ be the product manifold endowed with the product Poisson structure, that is, if $F, G : P_1 \times P_2 \to \mathbb{R}$, then

$$\{F, G\}(z_1, z_2) = \{F_{z_2}, G_{z_2}\}_1(z_1) + \{F_{z_1}, G_{z_1}\}_2(z_2),$$

where $\{\, ,\}_i$ is the Poisson bracket on P_i, $F_{z_1} : P_2 \to \mathbb{R}$ is the function obtained by freezing $z_1 \in P_1$, and similarly for $F_{z_2} : P_1 \to \mathbb{R}$. Let the Lie algebra $\mathfrak{g}$ act canonically on P_1 and P_2 with (equivariant) momentum mappings $\mathbf{J}_1 : P_1 \to \mathfrak{g}^*$ and $\mathbf{J}_2 : P_2 \to \mathfrak{g}^*$. Then

$$\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2 : P_1 \times P_2 \to \mathfrak{g}^*, \ \mathbf{J}(z_1, z_2) = \mathbf{J}(z_1) + \mathbf{J}(z_2)$$

is an (equivariant) momentum mapping of the canonical $\mathfrak{g}$-action on the product $P_1 \times P_2$. There is an obvious generalization to the product of N Poisson manifolds. Note that Example (b) is a special case of this, for $G = \mathbb{R}^3$ for all factors in the product manifold equal to $T^*\mathbb{R}^3$. ♦

(h) Cotangent Lift on T^*G. The momentum map for the cotangent lift of the *left* translation action of G on G is, by (12.1.14), equal to

$$\langle \mathbf{J}_L(\alpha_g), \xi \rangle = \langle \alpha_g, \xi_G(g) \rangle = \langle \alpha_g, T_e R_g(\xi) \rangle = \langle T_e^* R_g(\alpha_g), \xi \rangle \,,$$

that is,

$$\mathbf{J}_L(\alpha_g) = T_e^* R_g(\alpha_g). \tag{12.2.6}$$

Similarly, the momentum map for the lift to T^*G of *right* translation of G on G equals

$$\mathbf{J}_R(\alpha_g) = T_e^* L_g(\alpha_g). \tag{12.2.7}$$

Notice that $\mathbf{J}_L$ is *right* invariant, whereas $\mathbf{J}_R$ is *left* invariant. Both are equivariant momentum maps ($\mathbf{J}_R$ with respect to Ad_g^*, which is a *right* action), so they are Poisson maps. The diagram in Figure 12.3.1 summarizes the situation.

This diagram is an example of what is called a ***dual pair***; these illuminate the relation between the body and spatial description of rigid bodies and fluids; see Chapter 15 for more information. ♦

(j) Momentum Translation on Functions. Let $P = \mathcal{F}(T^*Q)^*$ with the Lie–Poisson bracket given in Example (e) of §10.2. Using the Liouville measure on T^*Q and assuming that elements of $\mathcal{F}(T^*Q)$ fall off rapidly

$$\begin{tikzcd}
 & T^*G \arrow[dl, "{\begin{array}{l}\mathbf{J}_L = \\ \text{right translation to } e\end{array}}"'] \arrow[dr, "{\begin{array}{l}\mathbf{J}_R = \\ \text{left translation to } e\end{array}}"] & \\
\mathfrak{g}^*_+ & & \mathfrak{g}^*_-
\end{tikzcd}$$

FIGURE 12.2.1. Momentum maps for left and right translations.

enough at infinity, we identify $\mathcal{F}(T^*Q)^*$ with $\mathcal{F}(T^*Q)$ using the L^2-pairing. Let $G = \mathcal{F}(Q)$ act on P by

$$(\varphi \cdot f)(\alpha_q) = f(\alpha_q + \mathbf{d}\varphi(q)), \tag{12.2.8}$$

that is, in coordinates,

$$f(q^i, p_j) \mapsto f\left(q^i, p_j + \frac{\partial \varphi}{\partial q^i}\right).$$

The infinitessimal generator is

$$\xi_P(f)(\alpha_q) = \mathbb{F}f(\alpha_q) \cdot \mathbf{d}\xi(q), \tag{12.2.9}$$

where $\mathbb{F}f$ is the fiber derivative of f. In coordinates, (12.2.9) reads

$$\xi_P(f)(q^i, p_j) = \frac{\partial f}{\partial p_j} \cdot \frac{\partial \xi}{\partial q^j}.$$

As usual, we assume that all elements of $P = \mathcal{F}(T^*Q)$ fall off at infinity. Then, if $f, g, h \in \mathcal{F}(T^*Q)$ we have by Corollary 5.5.8

$$\int_{T^*Q} f\{g, h\}\, dq\, dp = \int_{T^*Q} g\{h, f\}\, dq\, dp. \tag{12.2.10}$$

Next, note that if $F, H : P = \mathcal{F}(T^*Q) \to \mathbb{R}$, then we get by (12.2.10)

$$\begin{aligned} X_H[F](f) = \{F, H\}(f) &= \int_P f \left\{\frac{\delta F}{\delta f}, \frac{\delta H}{\delta f}\right\} dq\, dp \\ &= \int_P \frac{\delta F}{\delta f} \left\{\frac{\delta H}{\delta f}, f\right\} dq\, dp. \end{aligned}$$

On the other hand, by (12.2.9), we have

$$\xi_P[F](f) = \int_P \frac{\delta F}{\delta f} (\mathbb{F}f \cdot (\mathbf{d}\xi \circ \pi_Q))\, dq\, dp, \tag{12.2.11}$$

which suggests that the definition of $\mathbf{J}$ should be

$$\langle \mathbf{J}(f), \xi\rangle = \int_P f(\alpha_q)\xi(q)\, dq\, dp. \tag{12.2.12}$$

Indeed, by (12.2.12), we have $\delta J(\xi)/\delta f = \xi \circ \pi_Q$ so that

$$\left\{ \frac{\delta J(\xi)}{\delta f}, f \right\} = \{\xi \circ \pi_Q, f\} = \mathbb{F}f \cdot (\mathbf{d}\xi \circ \pi_Q)$$

and hence by (12.2.10)

$$\begin{aligned} X_{J(\xi)}[F](f) &= \int_P \frac{\delta F}{\delta f} \left\{ \frac{\delta J(\xi)}{\delta f}, f \right\} dq\, dp \\ &= \int_P \frac{\delta F}{\delta f} (\mathbb{F}f \cdot (\mathbf{d}\xi \circ \pi_Q))\, dq\, dp, \end{aligned}$$

which coincides with (12.2.11) thereby proving that $\mathbf{J}$ given by (12.2.12) is the momentum map. In other words, the fiber integral

$$\mathbf{J}(f) = \int_P f(q,p)\, dp \tag{12.2.13}$$

is the momentum map in this case. This momentum map is infinitessimally equivariant. Indeed, if $\xi, \eta \in \mathcal{F}(Q)$, we have for $f \in P$,

$$\begin{aligned} \{J(\xi), J(\eta)\}(f) &= \int_P f \left\{ \frac{\delta J(\xi)}{\delta f}, \frac{\delta J(\eta)}{\delta f} \right\} dq\; dp \\ &= \int_P f\, \{\xi \circ \pi_Q, \eta \circ \pi_Q\}\, dq\; dp \\ &= 0 \;=\; J([\xi, \eta])(f). \end{aligned}$$

♦

changed 'denote' to 'be'

(j) More Momentum Translations. Let $\mathrm{Diff}_{\mathrm{can}}(T^*Q)$ be the group of symplectic diffeomorphisms of T^*Q and, as above, let $G = \mathcal{F}(Q)$ act on T^*Q by translation with $\mathbf{d}f$ along the fiber, that is, $f \cdot \alpha_q = \alpha_q + \mathbf{d}f(q)$. Since the action of the additive group $\mathcal{F}(Q)$ is Hamiltonian, $\mathcal{F}(Q)$ acts on $\mathrm{Diff}_{\mathrm{can}}(T^*Q)$ by composition on the right with translations, that is, the action is $(f, \varphi) \in \mathcal{F}(Q) \times \mathrm{Diff}_{\mathrm{can}}(T^*Q) \mapsto \varphi \circ \rho_f \in \mathrm{Diff}_{\mathrm{can}}(T^*Q)$, where $\rho_f(\alpha_q) = \alpha_q + \mathbf{d}f(q)$. The infinitessimal generator of this action is given by (see (12.1.17)):

$$\xi_{\mathrm{Diff}_{\mathrm{can}}(T^*Q)}(\varphi) = -T\varphi \circ X_{\xi \circ \pi_Q} \tag{12.2.14}$$

for $\xi \in \mathcal{F}(Q) = \mathfrak{g}$, so that the equivariant momentum map of the lifted action $\mathbf{J} : T^*(\mathrm{Diff}_{\mathrm{can}}(T^*Q)) \to \mathcal{F}(Q)^*$ given by (12.1.14) in this case

$$J(\xi)(\alpha_\varphi) = -\left\langle \alpha_\varphi, T\varphi \circ X_{\xi \circ \pi_Q} \right\rangle, \tag{12.2.15}$$

where the pairing on the right is between vector fields and one-form densities α_φ. ♦

(k) Maxwell's Equations. Let $\mathcal{A}$ be the space of vector potentials $\mathbf{A}$ on $\mathbb{R}^3$ and $P = T^*\mathcal{A}$, whose elements are denoted $(\mathbf{A}, -\mathbf{E})$ with $\mathbf{A}$ and $\mathbf{E}$ vector fields. Let $G = \mathcal{F}(\mathbb{R}^3)$ act on $\mathcal{A}$ by $\varphi \cdot \mathbf{A} = \mathbf{A} + \nabla\varphi$. Thus, the infinitessimal generator is

$$\xi_{\mathcal{A}}(\mathbf{A}) = \nabla\xi.$$

Hence the momentum map is

$$\langle \mathbf{J}(\mathbf{A}, -\mathbf{E}), \xi\rangle = \int -\mathbf{E} \cdot \nabla\xi \, d^3x = \int (\operatorname{div} \mathbf{E})\xi \, d^3x \tag{12.2.16}$$

(assuming fast enough falloff to justify integration by parts). Thus,

$$\mathbf{J}(\mathbf{A}, -\mathbf{E}) = \operatorname{div} \mathbf{E} \tag{12.2.17}$$

is the equivariant momentum map. ♦

(l) Virtual Work. We usually think of covectors as momenta conjugate to configuration variables. However, covectors can also be thought of as forces. Indeed, if $\alpha_q \in T_q^*Q$ and $w_q \in T_qQ$, we think of

$$\langle \alpha_q, w_q\rangle = \text{force} \times \text{infinitessimal displacement}$$

as the ***virtual work***. We now give an example of a momentum map in this context.

Consider a region $\mathcal{B} \subset \mathbb{R}^3$ with boundary $\partial\mathcal{B}$. Let $\mathcal{C}$ be the space of maps $\varphi : \mathcal{B} \to \mathbb{R}^3$. Regard $T_\varphi^*\mathcal{C}$ as the space of ***loads***; that is, pairs of maps $\mathbf{b} : \mathcal{B} \to \mathbb{R}^3$, $\tau : \partial\mathcal{B} \to \mathbb{R}^3$ paired with a tangent vector $\mathbf{V} \in T_\varphi\mathcal{C}$ by

$$\langle (\mathbf{b}, \tau), \mathbf{V}\rangle = \iiint_{\mathcal{B}} \mathbf{b} \cdot \mathbf{V} \, d^3x + \iint_{\partial\mathcal{B}} \tau \cdot \mathbf{V} \, dA.$$

Let $\mathbf{A} \in \mathrm{GL}(3, \mathbb{R})$ act on $\mathcal{C}$ by $\varphi \mapsto \mathbf{A} \circ \varphi$. The infinitessimal generator of this action is $\xi_{\mathcal{C}}(\varphi)(X) = \xi\varphi(X)$ for $\xi \in \mathfrak{gl}(3)$ and $X \in \mathcal{B}$. Pair $\mathfrak{gl}(3, \mathbb{R})$ with itself via $\langle \mathbf{A}, \mathbf{B}\rangle = \frac{1}{2}\operatorname{tr}(\mathbf{AB})$. The induced momentum map $\mathbf{J} : T^*\mathcal{C} \to \mathfrak{gl}(3, \mathbb{R})$ is given by

$$\mathbf{J}(\varphi, (\mathbf{b}, \tau)) = \iiint_{\mathcal{B}} \varphi \otimes \mathbf{b} \, d^3x + \iint_{\partial\mathcal{B}} \varphi \otimes \tau \, dA. \tag{12.2.18}$$

(This is the "astatic load," a concept from elasticity; see, for example, Marsden and Hughes [1983].) If we take $\mathrm{SO}(3)$ rather than $\mathrm{GL}(3, \mathbb{R})$, we get the angular momentum. ♦

(m) Momentum Maps for Unitary Representations on Projective Space.

Here we show that the momentum map discussed in Example g of §11.5 is equivariant. Recall from the discussion at the end of §9.3 that associated to a unitary representation ρ of a Lie group G on a complex Hilbert space $\mathcal{H}$, there are skew adjoint operators $A(\xi)$ for each $\xi \in \mathfrak{g}$ depending linearly on ξ and such that $\rho(\exp(t\xi)) = \exp(tA(\xi))$. Thus, taking the t-derivative in the formula

$$\rho(g)\rho(\exp(t\xi))\rho(g^{-1}) = \exp(t\rho(g)A(\xi)\rho(g)^{-1}),$$

we get

$$A(\mathrm{Ad}_g \xi) = \rho(g)A(\xi)\rho(g)^{-1}. \tag{12.2.19}$$

Using the formula we derived in §11.5, namely

$$\langle \mathbf{J}([\psi]), \xi \rangle = J(\xi)([\psi]) = -i\hbar \frac{\langle \psi, A(\xi)\psi \rangle}{\|\psi\|^2}, \tag{12.2.20}$$

we get

$$\begin{aligned} J(\mathrm{Ad}_g \xi)([\psi]) &= -i\hbar \frac{\langle \psi, \rho(g)A(\xi)\rho(g)^{-1}\psi \rangle}{\|\psi\|^2} \\ &= J(\xi)([\rho(g)^{-1}\psi]) = J(\xi)(g^{-1} \cdot [\psi]), \end{aligned}$$

which shows that $\mathbf{J} : \mathbb{P}\mathcal{H} \to \mathfrak{g}^*$ is equivariant. ♦

Exercises

◇ **Exercise 12.2-1.** Derive the conservation of $\mathbf{J}$ given by

$$\langle \mathbf{J}(v_q), \xi \rangle = \langle \mathbb{F}L(v_q), \xi_Q(q) \rangle$$

directly from Hamilton's variational principle. (This is the way Noether originally derived conserved quantities).

◇ **Exercise 12.2-2.** If L is independent of one of the coordinates q^i, then it is clear that $p_i = \partial L/\partial \dot{q}^i$ is a constant of the motion from the Euler–Lagrange equations. Derive this from Proposition 12.2.1.

◇ **Exercise 12.2-3.** Compute $\mathbf{J}_L$ and $\mathbf{J}_R$ for $G = \mathrm{SO}(3)$.

◇ **Exercise 12.2-4.** Compute the momentum maps determined by spatial translations and rotations for Maxwell's equations.

◇ **Exercise 12.2-5.** Repeat Exercise 12.2-4 for elasticity (the context of Example (m)).

check this exercise no., it was 12.3-2 in the earlier version before the Examples section was removed

◇ **Exercise 12.2-6.** Let P be a symplectic manifold and $\mathbf{J} : P \to \mathfrak{g}^*$ be an (equivariant) momentum map for the symplectic action of a group G on P. Let $\mathcal{F}$ be the space of (smooth) functions on P identified with its dual via integration and equipped with the Lie–Poisson bracket. Let $\mathcal{J} : \mathcal{F} \to \mathfrak{g}^*$ be defined by

$$\langle \mathcal{J}(f), \xi) \rangle = \int f \, \langle \mathbf{J}, \xi \rangle \, d\mu,$$

where μ is Liouville measure. Show that $\mathcal{J}$ is an (equivariant) momentum map.

$\diamond$ **Exercise 12.2-7.** Let G act on itself by conjugation. Compute the momentum map of its cotangent lift.

12.3 Equivariance and Infinitesimal Equivariance

This optional section explores the equivariance of momentum maps a little deeper. We have just seen that equivariance implies infinitessimal equivariance. In this section, we prove, amongst other things, the converse if G and P are connected.

A Family of Casimir Functions. Introduce the map $\Gamma_\eta : G \times P \to \mathbb{R}$ defined by

$$\Gamma_\eta(g, z) = \langle \mathbf{J}(\Phi_g(z)), \eta \rangle - \left\langle \operatorname{Ad}^*_{g^{-1}} \mathbf{J}(z), \eta \right\rangle \quad \text{for } \eta \in \mathfrak{g}. \tag{12.3.1}$$

Since

$$\Gamma_{\eta,g}(z) := \Gamma_\eta(g, z) = \left(\Phi_g^* J(\eta)\right)(z) - J\left(\operatorname{Ad}_{g^{-1}} \eta\right)(z), \tag{12.3.2}$$

we get

$$\begin{aligned} X_{\Gamma_{\eta,g}} &= X_{\Phi_g^* J(\eta)} - X_{J\left(\operatorname{Ad}_{g^{-1}} \eta\right)} \\ &= \Phi_g^* X_{J(\eta)} - \left(\operatorname{Ad}_{g^{-1}} \eta\right)_P \\ &= \Phi_g^* \eta_P - \left(\operatorname{Ad}_{g^{-1}} \eta\right)_P = 0 \end{aligned} \tag{12.3.3}$$

by (11.1.4). Therefore, $\Gamma_{\eta,g}$ is a Casimir function on P, and so is constant on every symplectic leaf of P. Since $\eta \mapsto \Gamma_\eta(g, z)$ is linear for every $g \in G$ and $z \in P$, we can define the map $\sigma : G \to L(\mathfrak{g}, \mathcal{C}(P))$, from G to the vector space of all linear maps of $\mathfrak{g}$ into the space of Casimir functions $\mathcal{C}(P)$ on P, by $\sigma(g) \cdot \eta = \Gamma_{\eta,g}$. The behavior of σ under group multiplication is the following. For $\xi \in \mathfrak{g}$, $z \in P$, and $g, h \in G$, we have

$$\begin{aligned}
(\sigma(gh) \cdot \xi)(z) &= \Gamma_\xi(gh, z) \\
&= (\mathbf{J}(\Phi_{gh}(z)), \xi) - \left\langle \mathrm{Ad}^*_{(gh)^{-1}} \mathbf{J}(z), \xi \right\rangle \\
&= \langle \mathbf{J}(\Phi_g(\Phi_h(z))), \xi \rangle - \left\langle \mathrm{Ad}^*_{g^{-1}} \mathbf{J}((\Phi_h(z)), \xi \right\rangle \\
&\quad + \left\langle \mathbf{J}(\Phi_h(z)), \mathrm{Ad}_{g^{-1}} \xi \right\rangle - \left\langle \mathrm{Ad}^*_{h^{-1}} \mathbf{J}(z), \mathrm{Ad}_{g^{-1}} \xi \right\rangle \\
&= \Gamma_\xi(g, \Phi_h(z)) + \Gamma_{\mathrm{Ad}_{g^{-1}} \xi}(h, z) \\
&= (\sigma(g) \cdot \xi)(\Phi_h(z)) + \left(\sigma(h) \cdot \mathrm{Ad}_{g^{-1}} \xi\right)(z). \qquad (12.3.4)
\end{aligned}$$

Connected Lie group actions admitting momentum maps preserve symplectic leaves. This is because G is generated by a neighborhood of the identity in which each element has the form $\exp t\xi$; since $(t, z) \mapsto (\exp t\xi) \cdot z$ is a Hamiltonian flow, it follows that z and $\Phi_h(z)$ are on the same leaf. Thus,

$$(\sigma(g) \cdot \xi)(z) = (\sigma(g) \cdot \xi)(\Phi_h(z))$$

because Casimir functions are constant on leaves. Therefore,

$$\sigma(gh) = \sigma(g) + \mathrm{Ad}^\dagger_{g^{-1}} \sigma(h), \qquad (12.3.5)$$

where $\mathrm{Ad}^\dagger_g$ denotes the action of G on $L(\mathfrak{g}, \mathcal{C}(P))$ induced via the adjoint action by

$$(\mathrm{Ad}^\dagger_g \lambda)(\xi) = \lambda(\mathrm{Ad}_{g^{-1}} \xi) \qquad (12.3.6)$$

for $g \in G$, $\xi \in \mathfrak{g}$, and $\lambda \in L(\mathfrak{g}, \mathcal{C}(P))$.

Cocycles. Mappings $\sigma : G \to L(\mathfrak{g}, \mathcal{C}(P))$, behaving under group multiplication as in (12.3.5), are called $L(\mathfrak{g}, \mathcal{C}(P))$-valued ***one-cocycles*** of the group G. A one-cocycle σ is called a ***one-coboundary*** if there is a $\lambda \in L(\mathfrak{g}, \mathcal{C}(P))$ such that

$$\sigma(g) = \lambda - \mathrm{Ad}^\dagger_{g^{-1}} \lambda \quad \text{for all } g \in G. \qquad (12.3.7)$$

The quotient space of one-cocycles modulo one-coboundaries is called the ***first*** $L(\mathfrak{g}, \mathcal{C}(P))$***-valued group cohomology of*** G and is denoted by $H^1(G, L(\mathfrak{g}, \mathcal{C}(P)))$; its elements are denoted by $[\sigma]$, for σ a one-cocycle.

At the Lie algebra level, bilinear skew-symmetric maps $\Sigma : \mathfrak{g} \times \mathfrak{g} \to \mathcal{C}(P)$ satisfying the Jacobi type identity (11.6.1) are called $\mathcal{C}(P)$***-valued two-cocycles of*** $\mathfrak{g}$. A cocycle Σ is called a ***coboundary*** if there is a $\lambda \in L(\mathfrak{g}, \mathcal{C}(P))$ such that

$$\Sigma(\xi, \eta) = \lambda([\xi, \eta]) \quad \text{for all } \xi, \eta \in \mathfrak{g}. \qquad (12.3.8)$$

The quotient space of two-cocycles by two-coboundaries is called the ***second cohomology of*** $\mathfrak{g}$ ***with values in*** $\mathcal{C}(P)$. It is denoted by $H^2(\mathfrak{g}, \mathcal{C}(P))$ and its elements by $[\Sigma]$. With these notations we have proved the first two parts of the following proposition:

Proposition 12.3.1. *Let the connected Lie group G act canonically on the Poisson manifold P and have a momentum map $\mathbf{J}$. For $g \in G$ and $\xi \in \mathfrak{g}$, define*

$$\Gamma_{\xi,g} : P \to \mathbb{R}, \quad \Gamma_{\xi,g}(z) = \langle \mathbf{J}\left(\Phi_g(z)\right), \xi\rangle - \left\langle \mathrm{Ad}^*_{g^{-1}}\ \mathbf{J}(z), \xi\right\rangle. \tag{12.3.9}$$

Then

(i) *$\Gamma_{\xi,g}$ is a Casimir on P for every $\xi \in \mathfrak{g}$ and $g \in G$.*

(ii) *Defining $\sigma : G \to L(\mathfrak{g}, \mathcal{C}(P))$ by $\sigma(g) \cdot \xi = \Gamma_{\xi,g}$, we have the identity*

$$\sigma(gh) = \sigma(g) + \mathrm{Ad}^\dagger_{g^{-1}}\ \sigma(h). \tag{12.3.10}$$

(iii) *Defining $\sigma_\eta : G \to \mathcal{C}(P)$ by $\sigma_\eta(g) := \sigma(g) \cdot \eta$ for $\eta \in \mathfrak{g}$, we have*

$$T_e\sigma_\eta(\xi) = \Sigma(\xi, \eta) := J([\xi, \eta]) - \{J(\xi), J(\eta)\}. \tag{12.3.11}$$

If $[\sigma] = 0$, then $[\Sigma] = 0$.

(iv) If $\mathbf{J}_1$ and $\mathbf{J}_2$ are two momentum mappings of the same action with cocycles σ_1 and σ_2, then $[\sigma_1] = [\sigma_2]$.

Proof. Since $\sigma_\eta(g)(z) = J(\eta)(g\cdot z) - J(\mathrm{Ad}_{g^{-1}}\ \eta)(z)$, taking the derivative at $g = e$, we get

$$\begin{aligned} T_e\sigma_\eta(\xi)(z) &= \mathbf{d}J(\eta)(\xi_P(z)) + J([\xi, \eta])(z) \\ &= X_{J(\xi)}[J(\eta)](z) + J([\xi, \eta])(z) \\ &= -\left\{J(\xi), J(\eta)\right\}(z) + J([\xi, \eta])(z). \end{aligned} \tag{12.3.12}$$

This proves (12.3.11). The second statement in (iii) is a consequence of the definition. To prove (iv) we note that

$$\sigma_1(g)(z) - \sigma_2(g)(z) = \mathbf{J}_1(g \cdot z) - \mathbf{J}_2(g \cdot z) - \mathrm{Ad}^*_{g^{-1}}(\mathbf{J}_1(z) - \mathbf{J}_2(z)). \tag{12.3.13}$$

However, $\mathbf{J}_1$ and $\mathbf{J}_2$ are momentum mappings of the same action and, therefore, $J_1(\xi)$ and $J_2(\xi)$ generate the same Hamiltonian vector field for all $\xi \in \mathfrak{g}$, so $J_1 - J_2$ is constant as an element of $L(\mathfrak{g}, \mathcal{C}(P))$. Calling this element λ, we have

$$\sigma_1(g) - \sigma_2(g) = \lambda - \mathrm{Ad}^\dagger_{g^{-1}}\ \lambda, \tag{12.3.14}$$

so $\sigma_1 - \sigma_2$ is a coboundary. ■

Remarks.

1. Part (iv) of this proposition also holds for Lie algebra actions admitting momentum maps with all σ's replaced by Σ's; indeed,

$$\{J_1(\xi), J_1(\eta)\} = \{J_2(\xi), J_2(\eta)\}$$

because $J_1(\xi) - J_2(\xi)$ and $J_1(\eta) - J_2(\eta)$ are Casimir functions.

2. If $[\Sigma] = 0$, *the momentum map* $\mathbf{J} : P \to \mathfrak{g}^*$ *of the canonical Lie algebra action of* $\mathfrak{g}$ *on* P *can be always chosen to be infinitessimally equivariant*, a result due to Souriau [1970] for the symplectic case. To see this, note first that momentum maps are determined only up to elements of $L(\mathfrak{g}, \mathcal{C}(P))$. Therefore, if $\lambda \in L(\mathfrak{g}, \mathcal{C}(P))$ denotes the element determined by the condition $[\Sigma] = 0$, then $J + \lambda$ is an infinitessimally equivariant momentum map.

3. *The cohomology class* $[\Sigma]$ *depends only on the Lie algebra action* $\rho : \mathfrak{g} \to \mathfrak{X}(P)$ *and not on the momentum map.* Indeed, because J is determined only up to the addition of a linear map $\lambda : \mathfrak{g} \to \mathcal{C}(P)$ and denoting

$$\Sigma_\lambda(\xi, \eta) := (J + \lambda)([\xi, \eta]) - \{(J + \lambda)(\xi), (J + \lambda)(\eta)\}\,, \tag{12.3.15}$$

we obtain

$$\begin{aligned}\Sigma_\lambda(\xi, \eta) &= J([\xi, \eta]) + \lambda([\xi, \eta]) - \{J(\xi), J(\eta)\}\\ &= \Sigma(\xi, \eta) + \lambda([\xi, \eta]),\end{aligned} \tag{12.3.16}$$

that is, $[\Sigma_\lambda] = [\Sigma]$. Letting $\rho' \in H^2(\mathfrak{g}, \mathcal{C}(P))$ denote this cohomology class, $\mathbf{J}$ is infinitessimally equivariant if and only if ρ' vanishes. There are some cases in which one can predict that ρ' is zero:

(a) *Assume* P *is symplectic and connected (so* $\mathcal{C}(P) = \mathbb{R}$*) and suppose that* $H^2(\mathfrak{g}, \mathbb{R}) = 0$. By the second Whitehead lemma (see Jacobson [1962] or Guillemin and Sternberg [1984]), this is the case whenever $\mathfrak{g}$ is semisimple; thus semisimple, symplectic Lie algebra actions on symplectic manifolds are Hamiltonian.

(b) *Suppose* P *is exact symplectic,* $-\mathbf{d}\theta = \Omega$*, and*

$$\pounds_{\xi_P} \theta = 0. \tag{12.3.17}$$

The proof of equivariance in this case is the following. Assume first that the Lie algebra $\mathfrak{g}$ has an underlying Lie group G which leaves θ invariant. Since $\left(\mathrm{Ad}_{g^{-1}}\, \xi\right)_P = \Phi_g^* \xi_P$, we get from (11.3.4)

$$\begin{aligned}J(\xi)(g \cdot z) &= (\mathbf{i}_{\xi_P} \theta)\,(g \cdot z)\\ &= \left(\mathbf{i}_{\left(\mathrm{Ad}_{g^{-1}}\, \xi\right)_P} \theta\right)(z)\\ &= J\left(\mathrm{Ad}_{g^{-1}}\, \xi\right)(z).\end{aligned} \tag{12.3.18}$$

The proof without the assumption of the existence of the group G is obtained by differentiating the above string of equalities with respect to g at $g = e$.

A simple example in which $\rho' \neq 0$ is provided by phase-space translations on $\mathbb{R}^2$ defined by $\mathfrak{g} = \mathbb{R}^2 = \{(a,b)\}$, $P = \mathbb{R}^2 = \{(q,p)\}$, and

$$(a,b)_P = a\frac{\partial}{\partial q} + b\frac{\partial}{\partial p}. \tag{12.3.19}$$

This action has a momentum map given by $\langle \mathbf{J}(q,p),(a,b)\rangle = ap - bq$ and

$$\begin{aligned}\Sigma\left(\left(a_1,b_1\right),\left(a_2,b_2\right)\right) &= J\left(\left[\left(a_1,b_1\right),\left(a_2,b_2\right)\right]\right)\\ &\qquad - \left\{J\left(a_1,b_1\right), J\left(a_2,b_2\right)\right\}\\ &= -\left\{a_1p - b_1q, a_2p - b_2q\right\}\\ &= b_1a_2 - a_1b_2.\end{aligned} \tag{12.3.20}$$

Since $[\mathfrak{g},\mathfrak{g}] = \{0\}$, the only coboundary is zero, so $\rho' \neq 0$. This example is amplified in Example (b) of §12.6.

4. *If P is symplectic and connected and σ is a one-cocycle of the G-action on P, then:*

(a) $g \cdot \mu = \mathrm{Ad}^*_{g^{-1}}\,\mu + \sigma(g)$ *is an action of G on $\mathfrak{g}^*$;* and

(b) $\mathbf{J}$ *is equivariant with respect to this action.*

Indeed, since P is symplectic and connected, $\mathcal{C}(P) = \mathbb{R}$, and thus $\sigma : G \to \mathfrak{g}^*$. By Proposition 12.4.1,

$$\begin{aligned}(gh)\cdot\mu &= \mathrm{Ad}^*_{(gh)^{-1}}\,\mu + \sigma(gh)\\ &= \mathrm{Ad}^*_{g^{-1}}\,\mathrm{Ad}^*_{h^{-1}}\,\mu + \sigma(g) + \mathrm{Ad}^*_{g^{-1}}\,\sigma(h)\\ &= \mathrm{Ad}^*_{g^{-1}}(h\cdot\mu) + \sigma(g) = g\cdot(h\cdot\mu),\end{aligned} \tag{12.3.21}$$

which proves (a); (b) is a consequence of the definition.

5. *If P is symplectic and connected,* $\mathbf{J} : P \to \mathfrak{g}^*$ *is a momentum map, and Σ is the associated real-valued Lie algebra two-cocycle, then the momentum map* $\mathbf{J}$ *can be explicitly adjusted to be infinitessimally equivariant by enlarging $\mathfrak{g}$ to the central extension defined by Σ.*

Indeed, the ***central extension defined by*** Σ is the Lie algebra $\mathfrak{g}' := \mathfrak{g} \oplus \mathbb{R}$ with the bracket given by

$$\left[\left(\xi,a\right),\left(\eta,b\right)\right] = \left(\left[\xi,\eta\right],\Sigma\left(\xi,\eta\right)\right). \tag{12.3.22}$$

Let $\mathfrak{g}'$ act on P by $\rho(\xi, a)(z) = \xi_P(z)$ and let $\mathbf{J}' : P \to (\mathfrak{g}')^* = \mathfrak{g}^* \oplus \mathbb{R}$ be the induced momentum map, that is, it satisfies

$$X_{J'(\xi,a)} = (\xi, a)_P = X_{J(\xi)}, \tag{12.3.23}$$

so that

$$J'(\xi, a) - J(\xi) = \ell(\xi, a), \tag{12.3.24}$$

where $\ell(\xi, a)$ is a constant on P and is linear in (ξ, a). Therefore,

$$\begin{aligned} &J'\left(\left[(\xi, a), (\eta, b)\right]\right) - \left\{J'(\xi, a), J'(\eta, a)\right\} \\ &\quad = J'\left([\xi, \eta], \Sigma(\xi, \eta)\right) - \{J(\xi) + \ell(\xi, a), J(\eta) + \ell(\eta, b)\} \\ &\quad = J\left([\xi, \eta]\right) + \ell\left([\xi, \eta], \Sigma(\xi, \eta)\right) - \{J(\xi), J(\eta)\} \\ &\quad = \Sigma(\xi, \eta) + \ell([(\xi, a), (\eta, b)]) \\ &\quad = (\lambda + \ell)([(\xi, a), (\eta, b)]), \end{aligned} \tag{12.3.25}$$

where $\lambda(\xi, a) = a$. Thus, the real-valued two-cocycle of the $\mathfrak{g}'$ action is a coboundary and hence J' can be adjusted to become infinitessimally equivariant. Thus,

$$J'(\xi, a) = J(\xi) - a \tag{12.3.26}$$

is the desired infinitessimally equivariant momentum map of $\mathfrak{g}'$ on P.

For example, the action of $\mathbb{R}^2$ on itself by translations has the nonequivariant momentum map $\langle \mathbf{J}(q,p), (\xi, \eta)\rangle = \xi p - \eta q$ with group one-cocycle $\sigma(x, y) \cdot (\xi, \eta) = \xi y - \eta x$; here we think of $\mathbb{R}^2$ endowed with the symplectic form $dq \wedge dp$. The corresponding infinitessimally equivariant momentum map of the central extension is given by (12.3.26), that is, by the expression

$$\langle \mathbf{J}'(q,p), (\xi, \eta, a)\rangle = \xi p - \eta q - a.$$

For more examples, see §12.6.

Consider the situation for the corresponding action of the central extension G' of G on P if $G = E$, a topological vector space regarded as an abelian Lie group. Then $\mathfrak{g} = E$, $T\sigma_\eta = \sigma_\eta$ by linearity of σ_η, so that $\Sigma(\xi, \eta) = \sigma(\xi) \cdot \eta$, with ξ on the right-hand side thought of as an element of the Lie group, G. One defines the central extension G' of G by the circle group S^1 as the Lie group having an underlying manifold $E \times S^1$, and whose multiplication is given by (Souriau [1969])

$$\left(q_1, e^{i\theta_1}\right) \cdot \left(q_2, e^{i\theta_2}\right) = \left(q_1 + q_2, \exp\left\{i\left[\theta_1 + \theta_2 + \tfrac{1}{2}\Sigma(q_1, q_2)\right]\right\}\right), \tag{12.3.27}$$

the identity element equal to $(0, 1)$, and the inverse given by

$$\left(q, e^{i\theta}\right)^{-1} = \left(-q, e^{-i\theta}\right).$$

Then the Lie algebra of G' is $\mathfrak{g}' = E \oplus \mathbb{R}$ with the bracket given by (12.3.22) and thus the G'-action on P given by $(q, e^{i\theta}) \cdot z = q \cdot z$ has an equivariant momentum map $\mathbf{J}$ given by (12.3.26). If $E = \mathbb{R}^2$, the group G' is the ***Heisenberg group*** (see Exercise 9.1-4). ♦

Global Equivariance. Assume J is a Lie algebra homomorphism. Since $\Gamma_{\eta,g}$ is a Casimir function on P for every $g \in G$ and $\eta \in \mathfrak{g}$, it follows that $\Gamma_\eta | G \times S$ is independent of $z \in S$, where S is a symplectic leaf. Denote this function that depends only on the leaf S by $\Gamma_\eta^S : G \to \mathbb{R}$. Fixing $z \in S$, and taking the derivative of the map $g \mapsto \Gamma_\eta^S(g, z)$ at $g = e$ in the direction $\xi \in \mathfrak{g}$, gives

$$\langle -(\mathrm{ad}\ \xi)^* \mathbf{J}(z), \eta \rangle - \langle T_z \mathbf{J} \cdot \xi_P(z), \eta \rangle = 0, \tag{12.3.28}$$

that is, $T_e \Gamma_\eta^S = 0$ for all $\eta \in \mathfrak{g}$. By Proposition 12.4.1(ii), we have

$$\Gamma_\eta(gh) = \Gamma_\eta(g) + \Gamma_{\mathrm{Ad}_{g^{-1}}\eta}(h). \tag{12.3.29}$$

Taking the derivative of (12.3.29) with respect to g in the direction ξ at $h = e$ on the leaf S and using $T_e \Gamma_\eta^S = 0$, we get

$$T_g \Gamma_\eta^S (T_e L_g(\xi)) = T_e \Gamma^S_{\mathrm{Ad}_{g^{-1}}\eta}(\xi) = 0. \tag{12.3.30}$$

Thus, Γ_η is constant on $G \times S$ (recall that symplectic leaves are, by definition, connected). Since $\Gamma_\eta(e, z) = 0$, it follows that $\Gamma_\eta | G \times S = 0$ for any leaf S and hence $\Gamma_\eta = 0$ on $G \times P$. But $\Gamma_\eta = 0$ for every $\eta \in \mathfrak{g}$ is equivalent to equivariance. Together with Theorem 11.6.1 this proves the following:

Theorem 12.3.2. *Let the connected Lie group G act canonically on the left on the Poisson manifold P. The action of G is globally Hamiltonian if and only if there is a Lie algebra homomorphism $\psi : \mathfrak{g} \to \mathcal{F}(P)$ such that $X_{\psi(\xi)} = \xi_P$ for all $\xi \in \mathfrak{g}$ where ξ_P is the infinitessimal generator of the G-action. If $\mathbf{J}$ is the equivariant momentum map of the action, then we can take $\psi = J$.*

The converse question of the construction of a group action whose momentum map equals a given set of conserved quantities closed under bracketing is addressed in Fong and Meyer [1975]. See also Vinogradov and Krasilshchick [1975] and Conn [1984], [1985] for the related question of when the germs of Poisson vector fields are Hamiltonian.

Exercises

◇ **Exercise 12.3-1.** Let G be a Lie group, its Lie algebra, and $\mathfrak{g}^*$ its dual. Let

$$\wedge^k(\mathfrak{g}^* = \{\alpha : \mathfrak{g}^* \times \cdots \times \mathfrak{g}^* \ (k \text{ times }) \to \mathbb{R} \mid \alpha \text{ is } k\text{-linear, } \alpha \text{ is skew-symmetric}\}$$

be the vector space of k-linear forms on $\mathfrak{g}^*$. Define

$$\mathbf{d} : \wedge^k(\mathfrak{g}^*) \longrightarrow \wedge^{k+1}(\mathfrak{g}^*), \qquad k \geq 1.$$

by

$$\mathbf{d}\alpha(\xi_0, \xi_1, \ldots \xi_k) = \sum_{0 \le i < j \le k} (-1)^{i+j} \alpha([\xi_i, \xi_j], \xi_0, \ldots, \hat{\xi}_i, \ldots \hat{\xi}_j, \ldots \xi_k),$$

where $\hat{\xi}_i$ means that ξ_i is omitted.

(a) Work out explicitly $\mathbf{d}\alpha$, if $\alpha \in \wedge^1(\mathfrak{g}^*)$ and $\alpha \in \wedge^2(\mathfrak{g}^*)$.

(b) Show that if we identify $\alpha \in \wedge^k(\mathfrak{g}^*)$ with its left invariant extension $\alpha_L \in \Omega^k(G)$ given by

$$\alpha_L(g)(v_1, \ldots v_k) = \alpha(T_e L_{g^{-1}} v_1, \ldots T_e L_{g^{-1}} v_k),$$

where $v_1, \ldots, v_k \in T_g G$, then $\mathbf{d}\alpha_L$ is the left invariant extension of $\mathbf{d}\alpha$, that is, $\mathbf{d}\alpha_L = (\mathbf{d}\alpha)_L$.

(c) Conclude that indeed $\mathbf{d}\alpha \in \wedge^{k+1}(\mathfrak{g}^*)$ if $\alpha \in \wedge^k(\mathfrak{g}^*)$ and that $\mathbf{d} \circ \mathbf{d} = 0$.

(d) Letting

$$Z^k(\mathfrak{g}) = \ker\left(\mathbf{d} : \wedge^k(\mathfrak{g}^*) \longrightarrow \wedge^{k+1}(\mathfrak{g}^*)\right)$$

be the subspace of *k-cocyles* and

$$B^k((\mathfrak{g}) = \text{range}\ \left(\mathbf{d} : \wedge^{k-1}(\mathfrak{g}^*) \longrightarrow \wedge^k(\mathfrak{g}^*)\right)$$

be the space of *k-coboundaries*, show that $B^k(\mathfrak{g}) \subset Z^k(\mathfrak{g})$. The quotient $H^k(\mathfrak{g})/B^k(\mathfrak{g})$ is the k**-th Lie algebra cohomology** group of $\mathfrak{g}$ with real coefficients.

12.4 Equivariant Momentum Maps Are Poisson

We next show that equivariant momentum maps are Poisson maps. This provides a fundamental method for finding canonical maps between Poisson manifolds. This result is partly contained in Lie's work [1890], is implicit in Guillemin and Sternberg [1980], and explicit in Holmes and Marsden [1983] and Guillemin and Sternberg [1984].

Theorem 12.4.1 (Canonical Momentum Maps). *If $\mathbf{J} : P \to \mathfrak{g}^*$ is an infinitessimally equivariant momentum map for a left Hamiltonian action of $\mathfrak{g}$ on a Poisson manifold P, then $\mathbf{J}$ is a Poisson map*:

$$\mathbf{J}^* \{F_1, F_2\}_+ = \{\mathbf{J}^* F_1, \mathbf{J}^* F_2\}, \tag{12.4.1}$$

that is,

$$\{F_1, F_2\}_+ \circ \mathbf{J} = \{F_1 \circ \mathbf{J}, F_2 \circ \mathbf{J}\}$$

for all $F_1, F_2 \in \mathcal{F}(\mathfrak{g}^*)$, where $\{\,,\}_+$ denotes the "+" Lie–Poisson bracket.

Proof. Infinitesimal equivariance means that $J([\xi,\eta]) = \{J(\xi), J(\eta)\}$. For $F_1, F_2 \in \mathcal{F}(\mathfrak{g}^*)$, let $z \in P$, $\xi = \delta F_1/\delta\mu$, and $\eta = \delta F_2/\delta\mu$ evaluated at the particular point $\mu = \mathbf{J}(z) \in \mathfrak{g}^*$. Then

$$\begin{aligned}
\mathbf{J}^* \{F_1, F_2\}_+ (z) &= \left\langle \mu, \left[\frac{\delta F_1}{\delta\mu}, \frac{\delta F_2}{\delta\mu}\right]\right\rangle \\
&= \langle \mu, [\xi,\eta]\rangle \\
&= J([\xi,\eta])(z) = \{J(\xi), J(\eta)\}(z).
\end{aligned}$$

But for any $z \in P$ and $v_z \in T_zP$,

$$\begin{aligned}
\mathbf{d}(F_1 \circ \mathbf{J})(z) \cdot v_z &= \mathbf{d}F_1(\mu) \cdot T_z\mathbf{J}(v_z) \\
&= \left\langle T_z\mathbf{J}(v_z), \frac{\delta F_1}{\delta\mu}\right\rangle \\
&= \mathbf{d}J(\xi)(z) \cdot v_z,
\end{aligned}$$

that is, $(F_1 \circ \mathbf{J})(z)$ and $J(\xi)(z)$ have equal z-derivatives. Since the Poisson bracket on P depends only on the point values of the first derivatives, we conclude that

$$\{F_1 \circ \mathbf{J}, F_2 \circ \mathbf{J}\}(z) = \{J(\xi), J(\eta)\}(z). \qquad \blacksquare$$

Theorem 12.4.2 (Collective Hamiltonian Theorem). *Let $\mathbf{J} : P \to \mathfrak{g}^*$ be a momentum map. Let $z \in P$ and $\mu = \mathbf{J}(z) \in \mathfrak{g}^*$. Then for any $F \in \mathcal{F}(\mathfrak{g}^*_+)$,*

$$X_{F\circ\mathbf{J}}(z) = X_{\mathbf{J}(\delta F/\delta\mu)}(z) = \left(\frac{\delta F}{\delta\mu}\right)_P (z). \tag{12.4.2}$$

Proof. For any $H \in \mathcal{F}(P)$,

$$\begin{aligned}
X_{F\circ\mathbf{J}}[H](z) &= -X_H[F\circ\mathbf{J}](z) \\
&= -\mathbf{d}(F\circ\mathbf{J})(z) \cdot X_H(z) \\
&= -\mathbf{d}F(\mu)(T_z\mathbf{J} \cdot X_H(z)) \\
&= -\left\langle T_z\mathbf{J}(X_H(z)), \frac{\delta F}{\delta\mu}\right\rangle \\
&= -\mathbf{d}J\left(\frac{\delta F}{\delta\mu}\right)(z) \cdot X_H(z) \\
&= -X_H\left[J\left(\frac{\delta F}{\delta\mu}\right)\right](z) \\
&= X_{J(\delta F/\delta\mu)}[H](z).
\end{aligned}$$

This proves the first equality in (12.4.2) and the second results from the definition of the momentum map. $\blacksquare$

Functions on P of the form $F \circ \mathbf{J}$ are called ***collective***. Note that if F is the linear function determined by $\xi \in \mathfrak{g}$, (12.4.2) reduces to $X_{J(\xi)}(z) = \xi_P(z)$, the definition of the momentum map. To demonstrate the relation between these results, let us derive Theorem 12.5.1 from Theorem 12.5.2. Let $\mu = \mathbf{J}(z)$, and $F, H \in \mathcal{F}(\mathfrak{g}^*_+)$. Then

$$\begin{aligned}
\mathbf{J}^* \left\{F, H\right\}_+ (z) &= \left\{F, H\right\}_+ (\mathbf{J}(z)) = \left\langle \mathbf{J}(z), \left[\frac{\delta F}{\delta \mu}, \frac{\delta H}{\delta \mu}\right]\right\rangle \\
&= J\left(\left[\frac{\delta F}{\delta \mu}, \frac{\delta H}{\delta \mu}\right]\right)(z) \;= \left\{J\left(\frac{\delta F}{\delta \mu}\right), J\left(\frac{\delta H}{\delta \mu}\right)\right\}(z) \\
&\hspace{14em}\text{(by infinitessimal equivariance)} \\
&= X_{J(\delta H/\delta \mu)}\left[J\left(\frac{\delta F}{\delta \mu}\right)\right](z) = X_{H \circ \mathbf{J}}\left[J\left(\frac{\delta F}{\delta \mu}\right)\right](z) \\
&\hspace{10em}\text{(by the collective Hamiltonian theorem)} \\
&= -X_{J(\delta F/\delta \mu)}[H \circ \mathbf{J}](z) = -X_{F \circ \mathbf{J}}[H \circ \mathbf{J}](z) \\
&\hspace{8em}\text{(again by the collective Hamiltonian theorem)} \\
&= \left\{F \circ \mathbf{J}, H \circ \mathbf{J}\right\}(z). \qquad \blacksquare
\end{aligned}$$

Remarks.

1. Let $i : \mathfrak{g} \to \mathcal{F}(\mathfrak{g}^*)$ denote the natural embedding of $\mathfrak{g}$ in its bidual; that is, $i(\xi) \cdot \mu = \langle \mu, \xi \rangle$. Since $\delta i(\xi)/\delta \mu = \xi$, i is a Lie algebra homomorphism, that is,

$$i([\xi, \eta]) = \left\{i(\xi), i(\eta)\right\}_+ . \tag{12.4.3}$$

We claim that *a canonical left Lie algebra action of $\mathfrak{g}$ on a Poisson manifold P is Hamiltonian if and only if there is a Poisson algebra homomorphism $\chi : \mathcal{F}(\mathfrak{g}^*_+) \to \mathcal{F}(P)$ such that $X_{(\chi \circ i)(\xi)} = \xi_P$ for all $\xi \in \mathfrak{g}$*. Indeed, if the action is Hamiltonian, let $\chi = \mathbf{J}^*$ (pull back on functions) and the assertion follows from the definition of momentum maps. The converse relies on the following fact. Let M, N be finite dimensional manifolds and $\chi : \mathcal{F}(N) \to \mathcal{F}(M)$ be a ring homomorphism. Then there exists a unique smooth map $\varphi : M \to N$ such that $\chi = \varphi^*$. (A similar statement holds for infinite-dimensional manifolds in the presence of some additional technical conditions. See Abraham, Marsden, and Ratiu [1988], Supplement 4.2C.) Therefore, if a ring and Lie algebra homomorphism $\mathcal{F}(\mathfrak{g}^*_+) \to \mathcal{F}(P)$ is given, there is a unique map $\mathbf{J} : P \to \mathfrak{g}^*$ such that $\chi = \mathbf{J}^*$. But for $\xi, \mu \in \mathfrak{g}^*$ we have

$$\begin{aligned}
[(\chi \circ i)(\xi)](z) &= \mathbf{J}^*(i(\xi))(z) = i(\xi)(\mathbf{J}(z)) \\
&= \langle \mathbf{J}(z), \xi \rangle = J(\xi)(z),
\end{aligned} \tag{12.4.4}$$

that is, $\chi \circ i = J$ which is a Lie algebra homomorphism because χ is, by hypothesis. Since $X_{J(\xi)} = \xi_P$ again by hypothesis, it follows that $\mathbf{J}$ is an infinitessimally equivariant momentum map.

2. Here we have worked with left actions. If in all statements one changes left by right actions and "+" by "−" in the Lie–Poisson structures on $\mathfrak{g}^*$, the resulting statements are true. ♦

Examples

(a) Phase Space Rotations. Let (P, Ω) be a linear symplectic space and let G be a subgroup of the linear symplectic group acting on P by matrix multiplication. The infinitessimal generator of $\xi \in \mathfrak{g}$ at $z \in P$ is

$$\xi_P(z) = \xi z, \tag{12.4.5}$$

where ξz is matrix multiplication. This vector field is Hamiltonian with Hamiltonian $\Omega(\xi z, z)/2$ by Proposition 2.7.1. Thus, a momentum map is

$$\langle \mathbf{J}(z), \xi \rangle = \tfrac{1}{2}\Omega(\xi z, z). \tag{12.4.6}$$

For $S \in G$, the adjoint action is

$$\mathrm{Ad}_S\, \xi = S\xi S^{-1}, \tag{12.4.7}$$

and hence

$$\begin{aligned} \left\langle \mathbf{J}(Sz), S\xi S^{-1} \right\rangle &= \frac{1}{2}\,\Omega(S\xi S^{-1}Sz, Sz) \\ &= \frac{1}{2}\,\Omega(S\xi z, Sz) = \frac{1}{2}\,\Omega(\xi z, z), \end{aligned} \tag{12.4.8}$$

so $\mathbf{J}$ is equivariant. Infinitesimal equivariance is a reformulation of (2.7.10). Notice that this momentum map is not of the cotangent lift type. ♦

(b) Phase Space Translations. Let (P, Ω) be a linear symplectic space and let G be a subgroup of the translation group of P, with $\mathfrak{g}$ identified with a linear subspace of P. Clearly

$$\xi_P(z) = \xi$$

in this case. The vector field is Hamiltonian with Hamiltonian given by the linear function

$$J(\xi)(z) = \Omega(\xi, z), \tag{12.4.9}$$

as is easily checked. This is therefore a momentum map for the action. This momentum map is not equivariant, however. The action of $\mathbb{R}^2$ on $\mathbb{R}^2$ by translation is a specific example; see the end of Remark 3 of §12.4. ♦

(c) Lifted Actions and Magnetic Terms. Another way nonequivariance of momentum maps comes up is with lifted cotangent actions, but with symplectic forms which are the canonical ones modified by the addition of a magnetic term. For example, endow $P = T^*\mathbb{R}^2$ with the symplectic form

$$\Omega_B = dq^1 \wedge dp_1 + dq^2 \wedge dp_2 + B\, dq^1 \wedge dq^2$$

where B is a function of q^1 and q^2. Consider the action of $\mathbb{R}^2$ on $\mathbb{R}^2$ by translations and lift this to an action of $\mathbb{R}^2$ on P. Note that this action preserves Ω_B if and only if B is constant, which will be assumed from now on. By (12.4.9) the momentum map is

$$\langle \mathbf{J}(\mathbf{q}, \mathbf{p}), \xi \rangle = \mathbf{p} \cdot \xi + B(\xi^1 q^2 - \xi^2 q^1). \tag{12.4.10}$$

This momentum map is not equivariant; in fact, since $\mathbb{R}^2$ is abelian, its Lie algebra two-cocycle is given by

$$\Sigma(\xi, \eta) = -\{J(\xi), J(\eta)\} = -2B(\xi^1\eta^2 - \xi^2\eta^1).$$

Let us assume from now on that B is nonzero. Viewed in different coordinates, the form Ω_B can be made canonical and the action by $\mathbb{R}^2$ is still translation by a canonical transformations. To do this, one switches to ***guiding center coordinates*** $(\mathbf{R}, \mathbf{P})$ defined by $\mathbf{P} = \mathbf{p}$ and $\mathbf{R} = (q^1 - p_2/B,\, q^2 + p_1/B)$. The physical interpretation of these coordinates is the following: $\mathbf{P}$ is the momentum of the particle, while $\mathbf{R}$ is the center of the nearly circular orbit pursued by the particle with coordinates $(\mathbf{q}, \mathbf{p})$ when the magnetic field is strong (Littlejohn [1983, 1984]). In these coordinates, Ω_B takes the form

$$\Omega_B = B dR^1 \wedge dR^2 - \frac{1}{B} dP_1 \wedge dP_2$$

and the $\mathbb{R}^2$-action on $T^*\mathbb{R}^2$ becomes translation in the $\mathbf{R}$-variable. The momentum map (12.4.10) becomes

$$\langle \mathbf{J}(\mathbf{R}, \mathbf{P}), \xi \rangle = B(\xi^1 R^2 - \xi^2 R^1) \tag{12.4.11}$$

which is again a special case of (12.2.5).

The cohomology class $[\Sigma] \neq 0$, as the following argument shows. If Σ was exact, there would exist a linear functional $\lambda : \mathbb{R}^2 \to \mathbb{R}$ such that $\Sigma(\xi, \eta) = \lambda([\xi, \eta]) = 0$ for all ξ, η; this is clearly false. Thus, $\mathbf{J}$ *cannot* be adjusted to obtain an equivariant momentum map.

Following Remark 5 of §12.4, the nonequivariance of the momentum map can be removed by passing to a central extension of $\mathbb{R}^2$. Namely, let $G' = \mathbb{R}^2 \times S^1$ with multiplication given by

$$(\mathbf{a}, e^{i\theta})(\mathbf{b}, e^{i\varphi}) = \left(\mathbf{a} + \mathbf{b}, e^{i(\theta + \varphi + B(a^1 b^2 - a^2 b^1))}\right) \tag{12.4.12}$$

and letting G' act on $T^*\mathbb{R}^2$ as before by

$$(\mathbf{a}, e^{i\theta}) \cdot (\mathbf{q}, \mathbf{p}) = (\mathbf{q} + \mathbf{a}, \mathbf{p}).$$

Then the momentum map $\mathbf{J} : T^*\mathbb{R}^2 \to \mathfrak{g}'^* = \mathbb{R}^3$ is given by

$$\langle \mathbf{J}(\mathbf{q}, \mathbf{p}), (\xi, a) \rangle = \mathbf{p} \cdot \xi + B(\xi^1 q^2 - \xi^2 q^1) - a. \tag{12.4.13}$$

♦

(d) Clairaut's Theorem. Let M be a surface of revolution in $\mathbb{R}^3$ obtained by revolving a graph $r = f(z)$ about the z-axis, where f is a smooth positive function. Pull back the usual metric of $\mathbb{R}^3$ to M and note that it is invariant under rotations about the z-axis. Consider the geodesic flow on M. The momentum map associated with the S^1 symmetry is $\mathbf{J} : TM \to \mathbb{R}$ given by $\langle \mathbf{J}(\mathbf{q}, \mathbf{v}), \xi \rangle = \langle (\mathbf{q}, \mathbf{v}), \xi_M(\mathbf{q}) \rangle$, as usual. Here, ξ_M is the vector field on $\mathbb{R}^3$ associated with a rotation with angular velocity ξ about the z-axis, so $\xi_M(\mathbf{q}) = \xi \mathbf{k} \times \mathbf{q}$. Thus,

$$\langle \mathbf{J}(\mathbf{q}, \mathbf{v}), \xi \rangle = \xi r \|\mathbf{v}\| \cos\theta,$$

where r is the distance to the z-axis and θ is the angle between $\mathbf{v}$ and the horizontal plane. Thus, as $\|\mathbf{v}\|$ is conserved, by conservation of energy, $r\cos\theta$ is *conserved along any geodesic on a surface of revolution, a statement known as* ***Clairaut's Theorem***. ♦

(e) Mass of a nonrelativistic free quantum particle. Here we show by means of an example, the relation between (genuine) projective unitary representations and non equivariance of the momentum map for the action on the projective space. This complements the discussion in Example (n) of §12.3 where we have shown that for unitary representations the momentum map is equivariant.

Let G be the Galilean group introduced in Remark 4 following XXXXX 9.3.9. Let $\mathcal{H} = L^2(\mathbb{R}^3)$ be the Hilbert space of square (Lebesgue) integrable complex functions on $\mathbb{R}^3$.

Fix a real number $m \neq 0$; for each $g = \{R, \mathbf{v}, \mathbf{a}, \tau\} \in G$, define the following unitary operator in $\mathcal{H}$:

$$(U_m(g)f)(\mathbf{p}) = \exp\left(i \left(\frac{\tau}{2m} |\mathbf{p}|^2 + (\mathbf{p} + m\mathbf{v}) \cdot \mathbf{a} \right) \right) f(R^{-1}(\mathbf{p} + m\mathbf{v})). \tag{12.4.14}$$

We can check by direct computation that:

$$U_m(g_1)U_m(g_2) = \exp(-im\sigma(g_1, g_2))U_m(g_1 g_2), \tag{12.4.15}$$

where (with $g_j = \{R_j, \mathbf{v}_j, \mathbf{a}_j, \tau_j\}$)

$$\sigma(g_1, g_2) = \tfrac{1}{2}|\mathbf{v}_1|^2 \tau_2 + (R_1 \mathbf{v}_2) \cdot (\mathbf{v}_1 \tau_2 + \mathbf{a}_1). \tag{12.4.16}$$

From (12.4.15) we see that the map $g \mapsto U_m(g)$ is not a group homomorphism, but when we project on $\mathrm{U}(\mathbb{P}\mathcal{H})$ (see Examples (d)-(h) after Definition 9.3.5) we get a group homomorphism, that is, a projective unitary representation of the Galilean group. Following the arguments in the examples quoted above and those in Proposition 5.3.1, we see that the Galilean group acts on $\mathbb{P}\mathcal{H}$ by symplectomorphisms. To get the momentum map, we notice that for any smooth $f \in \mathcal{H} = L^2(\mathbb{R}^3)$, the map $g \mapsto U_m(g) f \in \mathcal{H}$ is smooth, so it makes sense to define:

$$(a(\xi))f = \mathbf{d}(U_m(\cdot)f)(1)(\xi). \tag{12.4.17}$$

This shows that $a(\xi)$ is linear in ξ and defines a linear operator from $\mathcal{D} = C^\infty(\mathbb{R}^3)$ to $\mathcal{H} = L^2(\mathbb{R}^3)$. Because $U_m(g)$ is unitary and $U_m(1) = 1_{\mathcal{H}}$ it follows that $a(\xi)$ is formally skew adjoint on $\mathcal{D}$ for any $\xi \in \mathfrak{g}$. Explicitly, using the notations in Remark 4 after XXX 9.3.9, these operators are:

$$(a(\boldsymbol{\omega})f)(\mathbf{p}) = -\boldsymbol{\omega} \cdot \left(\mathbf{p} \times \frac{\partial f}{\partial \mathbf{p}}\right), \quad (a(\mathbf{u})f)(\mathbf{p}) = m\mathbf{u} \cdot \frac{\partial f}{\partial \mathbf{p}},$$

$$(a(\boldsymbol{\alpha})f)(\mathbf{p}) = i(\boldsymbol{\alpha} \cdot \mathbf{p})f(\mathbf{p}), \qquad (a(\theta)f)(\mathbf{p}) = i\theta \frac{|\mathbf{p}|^2}{2m} f(\mathbf{p}).$$

From these formulae we see that $\mathcal{D}$ is invariant under the group action and under $a(\xi)$ for each $\xi \in \mathfrak{g}$. From the theory of self adjoint operators one can show that $a(\xi)$ for each $\xi \in \mathfrak{g}$ is uniquely determined as a skew adjoint operator in $\mathcal{H}$. Therefore, $\mathbb{P}\mathcal{D}$ satisfies conditions (i)-(iii) of Example (f) following XXX 9.3.5 and is thus an essential G-smooth part of $\mathbb{P}\mathcal{H}$; on this set the momentum map can be defined. Following the computation in Example (g) of §11.5 we can write the momentum map for the action of the Galilean group on the projective space induced by the projective unitary representation (12.4.15):

$$J(\xi)([f]) = -\frac{i}{2} \frac{\langle f, a(\xi) f \rangle}{\|f\|^2} \tag{12.4.18}$$

for $f \neq 0$.

In spite of the fact that (12.4.18) and (11.4.23) look practically the same, the corresponding momentum maps have different properties because the infinitessimal generators $a(\xi)$ have different properties: in (11.4.23) $a(\xi)$ is uniquely determined by ξ, but here $a(\xi)$ is given by the projective representation only up to a linear functional on $\mathfrak{g}$. More crucial, the relation (12.2.18) which holds when the representation is unitary, may not be true for projective representations. In our case this relation becomes:

$$a(\mathrm{Ad}_g \xi) = U_m(g) a(\xi) U_m(g)^{-1} - 2i\Gamma_\xi(g^{-1}) 1_{\mathcal{H}}, \tag{12.4.19}$$

for some function Γ depending on $\xi \in \mathfrak{g}$ and $g \in G$. To show this, notice that from (12.4.15) we get:

$$U_m(\exp(ta(\mathrm{Ad}_g \xi)) = U_m(g) U_m(\exp(t\xi)) U_m(g)^{-1} \exp(im\gamma(g, t\xi)) \,,$$

with

$$\gamma(g, t\xi) = \sigma(g, \exp(t\xi)g^{-1}) + \sigma(\exp(t\xi), g^{-1}) - \sigma(g, g^{-1}).$$

Taking the derivative with respect to t at $t = 0$ and using (12.2.14) we get (12.2.16), where

$$\Gamma_\xi(g) = m\ \tfrac{1}{2}\frac{d}{dt}\gamma(g^{-1}, t\xi)\bigg|_{t=0}.$$

Using the notations in §9.3, we have for $\xi = \{\boldsymbol{\omega}, \mathbf{u}, \boldsymbol{\alpha}, \theta\}$ and $g = \{R, \mathbf{v}, \mathbf{a}, \tau\}$:

$$\Gamma_\xi(g) = -\frac{m}{2}\left(-\frac{1}{2}|\mathbf{v}|^2\theta + (\mathbf{a} \times \mathbf{v}) \cdot \omega + (\tau\mathbf{v} - \mathbf{a}) \cdot \mathbf{u} + \mathbf{v} \cdot \boldsymbol{\alpha}\right).$$

The corresponding Lie algebra cocycle as defined in (12.3.11) is given by:

$$\Sigma(\xi, \xi') = \frac{m}{2}(\mathbf{u} \cdot \boldsymbol{\alpha}' - \boldsymbol{\alpha} \cdot \mathbf{u}'). \tag{12.4.20}$$

This cocycle on the Lie algebra is nontrivial, that is, its cohomology class is non zero (see Exercise 12.6-3). Therefore, the mass of the particle measures the obstruction to equivariance for the momentum map (or for the projective representation to be a unitary representation) in $H^2(\mathfrak{g}, \mathbb{R})$. ♦

Exercises

◇ **Exercise 12.4-1.** Verify directly that angular momentum is a Poisson map.

◇ **Exercise 12.4-2.** What does the collective Hamiltonian theorem state for angular momentum? Is the result obvious?

◇ **Exercise 12.4-3.** If $z(t)$ is an integral curve of $X_{F\circ J}$, show that $\mu(t) = J(z(t))$ satisfies $\dot{\mu} = \mathrm{ad}^*_{\delta F/\delta\mu}\ \mu$.

◇ **Exercise 12.4-4.** Consider an ellipsoid of revolution in $\mathbb{R}^3$ and a geodesic starting at the "equator" making an angle of α with the equator. Use Clairaut's theorem to derive a bound on how high the geodesic climbs up the ellipse.

◇ **Exercise 12.4-5.** Consider the action of SE(2) on $\mathbb{R}^2$ as described in Exercise 11.5-2. Since this action was not defined as a lift, Theorem 12.1.4 is not applicable. In fact, in Exercise 11.6-2 it was shown that this momentum map is not equivariant. Compute the group and Lie algebra cocycles defined by this momentum map. Find the Lie algebra central extension making the momentum map equivariant.

◇ **Exercise 12.4-6.** Using Exercise 12.4-1, show that for the Galilean algebra, any 2-coboundary has the form:

$$\lambda(\xi, \xi') = \mathbf{x} \cdot (\boldsymbol{\omega} \times \boldsymbol{\omega}') + \mathbf{y} \cdot (\boldsymbol{\omega} \times \mathbf{u}' - \boldsymbol{\omega}' \times \mathbf{u}) + \mathbf{z} \cdot (\boldsymbol{\omega} \times \boldsymbol{\alpha}' - \boldsymbol{\omega}' \times \boldsymbol{\alpha} + u\theta' - u'\theta),$$

where

$$\xi = \{w, u\alpha, \theta\} \quad \text{and} \quad \xi' = \{w', u'\alpha', \theta'\}.$$

Conclude that the cocyle (12.2.17) is not a coboundary. (Actually, it can be proven that $H^2(\mathfrak{g}, \mathbb{R}) \cong \mathbb{R}$, that is, it is 1-dimensional, but this requires more algebraic work (Gullemin and Sternberg [1977,1984])).

◇ **Exercise 12.4-7.** Deduce the formula for the momentum map in Exercise 11.5-4 from (12.5.6)

`what/where is (12.5.6)?`

12.5 Poisson Automorphisms

Here are some miscellaneous facts about Poisson automorphisms, symplectic leaves, and momentum maps. For a Poisson manifold P, define the following Lie subalgebras of $\mathfrak{X}(P)$:

`Do you need numbers in the next 4 sections or will bullets be OK? Or alternatively change the next list to (i), (ii),... I changed the indented list to paragraphs to see if it looked better! --w`

1. ***Infinitesimal Poisson Automorphisms.*** Let $\mathcal{P}(P)$ be the set of $X \in \mathfrak{X}(P)$ such that:

$$X[\{F_1, F_2\}] = \{X[F_1], F_2\} + \{F_1, X[F_2]\}.$$

2. ***Infinitesimal Poisson Automorphisms Preserving Leaves.*** Let $\mathcal{PL}(P)$ be the set of $X \in \mathcal{P}(P)$ such that $X(z) \in T_zS$, where S is the symplectic leaf containing $z \in P$.

3. ***Locally Hamiltonian Vector Fields*** Let $\mathcal{LH}(P)$ be the set of $X \in \mathfrak{X}(P)$ such that for each $z \in P$, there is an open neighborhood U of z and an $F \in \mathcal{F}(U)$ such that $X|U = X_F|U$.

4. ***Hamiltonian Vector Fields.*** Let $\mathcal{H}(P)$ be the set of Hamiltonian vector fields X_F for $F \in \mathcal{F}(P)$.

Then one has the following facts (references are given if the verification is not straightforward):

1. $\mathcal{H}(P) \subset \mathcal{LH}(P) \subset \mathcal{PL}(P) \subset \mathcal{P}(P)$.

2. If P is symplectic, then $\mathcal{LH}(P) = \mathcal{PL}(P) = \mathcal{P}(P)$ and if $H^1(P) = 0$, then $\mathcal{LH}(P) = \mathcal{H}(P)$.

3. Let P be the trivial Poisson manifold, that is, $\{F, G\} = 0$ for all $F, G \in \mathcal{F}(P)$. Then $\mathcal{P}(P) \neq \mathcal{PL}(P)$.

4. Let $P = \mathbb{R}^2$ with the bracket
$$\{F, G\}(x, y) = x \left(\frac{\partial F}{\partial x} \frac{\partial G}{\partial y} - \frac{\partial F}{\partial x} \frac{\partial F}{\partial y} \right).$$
This is, in fact, a Lie–Poisson bracket. The vector field
$$X(x, y) = xy \frac{\partial}{\partial y}$$
is an example of an element of $\mathcal{PL}(P)$ which is not in $\mathcal{LH}(P)$.

5. $\mathcal{H}(P)$ is an ideal in any of the three Lie algebras including it. Indeed, if $Y \in \mathcal{P}(P)$ and $H \in \mathcal{F}(R)$, then $[Y, X_H] = X_{Y[H]}$.

6. If P is symplectic, then $[\mathcal{LH}(P), \mathcal{LH}(P)] \subset \mathcal{H}(P)$. (The Hamiltonian for $[X, Y]$ is $-\Omega(X, Y)$.) This is false for Poisson manifolds in general. If P is symplectic Calabi [1970] and Lichnerowicz [1973] showed that $[\mathcal{LH}(P), \mathcal{LH}(P)] = \mathcal{H}(P)$.

7. If the Lie algebra $\mathfrak{g}$ admits a momentum map on P, then $\mathfrak{g}_P \subset \mathcal{H}(P)$.

8. Let G be a connected Lie group. If the action admits a momentum map, it preserves the leaves of P. The proof was given in §12.4.

12.6 Momentum Maps and Casimir Functions

In this section we return to Casimir functions studied in Chapter 10 and link them with momentum maps. We will do this in the context of the Poisson manifolds P/G studied in §10.7.

We start with a Poisson manifold P and a free and proper Poisson action of a Lie group G on P admitting an equivariant momentum mapping $\mathbf{J} : P \to \mathfrak{g}^*$. We want to link $\mathbf{J}$ with a Casimir function $C : P/G \to \mathbb{R}$.

Proposition 12.6.1. *Let $\Phi : \mathfrak{g}^* \to \mathbb{R}$ be a function that is invariant under the coadjoint action. Then:*

(i) *Φ is a Casimir function for the Lie–Poisson bracket;*

(ii) *$\Phi \circ \mathbf{J}$ is G-invariant on P and so defines a function $C : P/G \to \mathbb{R}$ such that $\Phi \circ \mathbf{J} = C \circ \pi$, as in Figure 12.8.1; and*

$$\begin{tikzcd}
 & P \arrow[dl, "\pi"'] \arrow[dr, "\mathbf{J}"] & \\
P/G \arrow[dr, "C"'] & & \mathfrak{g}^* \arrow[dl, "\Phi"] \\
 & \mathbb{R} &
\end{tikzcd}$$

FIGURE 12.6.1. Casimir functions and momentum maps.

(iii) *the function C is a Casimir function on P/G.*

Proof. To prove the first part, we write down the condition of Ad^*-invariance as

$$\Phi(\mathrm{Ad}^*_{g^{-1}}\ \mu) = \Phi(\mu). \tag{12.6.1}$$

Differentiate this relation with respect to g at $g = e$ in the direction η to get by (9.3.2),

$$0 = \left.\frac{d}{dt}\right|_{t=0} \Phi\left(\mathrm{Ad}^*_{\exp(-t\eta)}\ \mu\right) = -\mathbf{D}\Phi(\mu)\cdot \mathrm{ad}^*_\eta\, \mu, \tag{12.6.2}$$

for all $\eta \in \mathfrak{g}$. Thus, by definition of $\delta\Phi/\delta\mu$,

$$0 = \left\langle \mathrm{ad}^*_\eta\ \mu, \frac{\delta\Phi}{\delta\mu}\right\rangle = \left\langle \mu, \mathrm{ad}_\eta\, \frac{\delta\Phi}{\delta\mu}\right\rangle = -\langle \mathrm{ad}^*_{\delta\Phi/\delta\mu}\ \mu, \eta\rangle$$

for all $\eta \in \mathfrak{g}$. In other words,

$$\mathrm{ad}^*_{\delta\Phi/\delta\mu}\ \mu = 0$$

so by Proposition 10.9.1, $X_\Phi = 0$ and thus Φ is a Casimir function.

To prove the second part, note that, by *equivariance* of $\mathbf{J}$ and *invariance* of Φ,

$$\Phi(\mathbf{J}(g\cdot z)) = \Phi(\mathrm{Ad}^*_{g^{-1}}\ \mathbf{J}(z)) = \Phi(\mathbf{J}(z)),$$

so $\Phi \circ \mathbf{J}$ is G-invariant.

Finally, for the third part, we use the collective Hamiltonian Theorem 12.5.2 to get for $\mu = \mathbf{J}(z)$,

$$X_{\Phi\circ\mathbf{J}}(z) = \left(\frac{\delta\Phi}{\delta\mu}\right)_P(z)$$

and so $T_z\pi \cdot X_{\Phi\circ\mathbf{J}}(z) = 0$ since infinitessimal generators are tangent to orbits, so project to zero under π. But π is Poisson, so

$$0 = T_z\pi\cdot X_{\Phi\circ\mathbf{J}}(z) = T_z\pi \cdot X_{C\circ\pi}(z) = X_C(\pi(z)).$$

Thus, C is a Casimir function on P/G. ■

Corollary 12.6.2. *If G is Abelian and $\Phi : \mathfrak{g}^* \to \mathbb{R}$ is any smooth function, then $\Phi \circ \mathbf{J} = C \circ \pi$ defines a Casimir function on P/G.*

This follows because for abelian groups, the Ad^*-action is trivial, so any function on $\mathfrak{g}^*$ is Ad^*-invariant.

Exercises

◇ **Exercise 12.6-1.** Verify that $\Phi(\Pi) = \|\Pi\|^2$ is an invariant function on $\mathfrak{so}(3)^*$.

◇ **Exercise 12.6-2.** Use Corollary 12.8.2 to find the Casimir functions for the bracket (10.7.6).