4

Interlude: Manifolds, Vector Fields, and Differential Forms

In preparation for later chapters, it will be necessary for the reader to learn a little bit about manifold theory. We recall a few basic facts here, beginning with the finite-dimensional case. (See Abraham, Marsden, and Ratiu [1988] for a full account.) The reader need not master all of this material now, but it suffices to read through it for general sense and come back to it repeatedly as our development of mechanics proceeds.

4.1 Manifolds

Coordinate Charts. Given a set M, a ***chart*** on M is an open set U in Euclidean space $\mathbb{R}^n$ with coordinates $(x^1, \ldots, x^n)$ (more generally U can be open in a Banach space) together with a one-to-one map $\varphi : U \to \varphi(U) \subset M$ of U onto some subset of M.

We call M a ***differentiable manifold*** if the following hold:

M1. *It is covered by a collection of charts, that is, every point is represented in at least one chart.*

M2. *If two charts U, U' have an overlapping image in M, then*

$$V = \varphi^{-1}(\varphi(U) \cap \varphi'(U')) \quad \textit{and} \quad V' = (\varphi')^{-1}(\varphi(U) \cap \varphi'(U'))$$

*are open sets in $\mathbb{R}^n$. Hence the mapping $\varphi'^{-1} \circ \varphi : V \to V'$ from an open subset of $\mathbb{R}^n$ to a subset of $\mathbb{R}^n$ is defined (Figure 4.1.1). The charts U, U' are called **compatible** if these n functions of n variables $\varphi'^{-1} \circ \varphi$ are C^∞.*

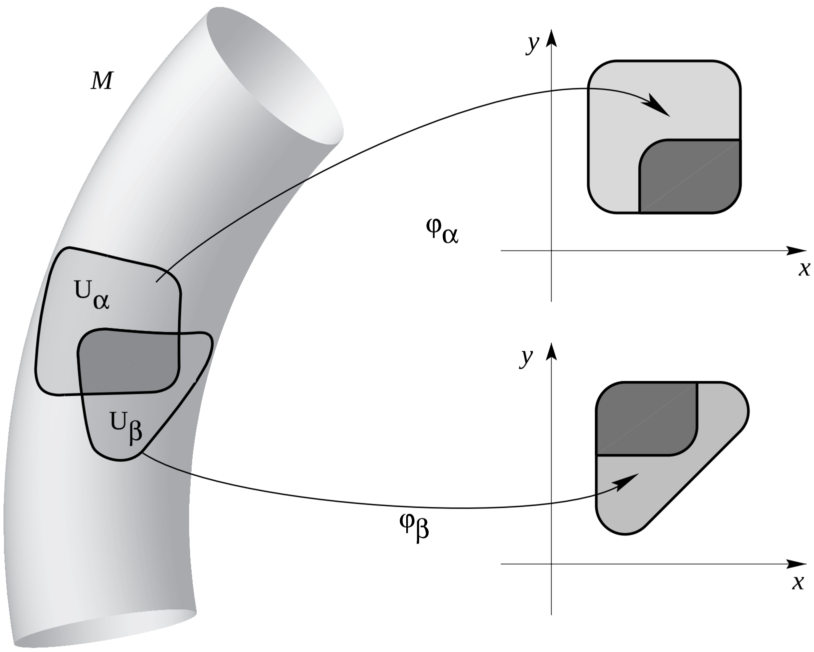

FIGURE 4.1.1. Overlapping charts on a manifold.

M3. *M has an* ***atlas****; that is, M can be written as a union of compatible charts.*

Two atlases are called ***equivalent*** if their union is also an atlas. One often rephrases the definition by saying that a differentiable structure on a manifold is an equivalence class of atlases.

A ***neighborhood*** of a point x in a manifold M is the image under a map $\varphi : U \to M$ of a neighborhood of the representation of x in a chart U. Neighborhoods define open sets and one checks that the open sets in M define a topology. *Usually we assume without explicit mention that the topology is Hausdorff*: two different points x, x' in M have nonintersecting neighborhoods. A differentiable manifold M is called an n-***manifold*** if every chart has domain in an n-dimensional vector space.

Another useful viewpoint is to think of M as a set covered by a collection of coordinate charts with local coordinates $(x^1, \ldots, x^n)$ with the property that all mutual changes of coordinates are smooth maps.

Tangent Vectors. Two curves $t \mapsto c_1(t)$ and $t \mapsto c_2(t)$ in an n-manifold M are called ***equivalent at*** x if

$$c_1(0) = c_2(0) = x \quad \text{and} \quad (\varphi^{-1} \circ c_1)'(0) = (\varphi^{-1} \circ c_2)'(0)$$

in some chart φ. It is easy to check that this definition is chart independent. A ***tangent vector*** v to a manifold M at a point $x \in M$ is an equivalence class of curves at x. One proves that the set of tangent vectors to M at x forms a vector space. It is denoted T_xM and is called the ***tangent space*** to M at $x \in M$. Given a curve $c(t)$, we denote by $c'(s)$ the tangent vector at $c(s)$ defined by the equivalence class of $t \mapsto c(s+t)$ at $t = 0$.

Let U be a chart of an atlas for the manifold M with coordinates $(x^1, \ldots, x^n)$. The ***components*** of the tangent vector v to the curve $t \mapsto (\varphi^{-1} \circ c)(t)$ are the numbers $v^1, \ldots, v^n$ defined by

$$v^i = \frac{d}{dt}(\varphi^{-1} \circ c)^i \bigg|_{t=0},$$

where $i = 1, \ldots, n$. The ***tangent bundle*** of M, denoted by TM, is the differentiable manifold whose underlying set is the disjoint union of the tangent spaces to M at the points $x \in M$, that is,

$$TM = \bigcup_{x \in M} T_xM.$$

Thus, a point of TM is a vector v that is tangent to M at some point $x \in M$. To define the differentiable structure on TM, we need to specify how to construct local coordinates on TM. To do this, let $x^1, \ldots, x^n$ be local coordinates on M and let $v^1, \ldots, v^n$ be components of a tangent vector in this coordinate system. Then the $2n$ numbers $x^1, \ldots, x^n, v^1, \ldots, v^n$ give a local coordinate system on TM. Notice that $\dim TM = 2 \dim M$.

The ***natural projection*** is the map $\tau_M : TM \to M$ that takes a tangent vector v to the point $x \in M$ at which the vector v is attached (that is, $v \in T_xM$). The inverse image $\tau_M^{-1}(x)$ of a point $x \in M$ under the natural projection τ_M is the tangent space T_xM. This space is called the ***fiber*** of the tangent bundle over the point $x \in M$.

Differentiable Maps. Let $f : M \to N$ be a map of a manifold M to a manifold N. We call f ***differentiable*** (or C^k) if in local coordinates on M and N it is given by differentiable (or C^k) functions. The ***derivative*** of a differentiable map $f : M \to N$ at a point $x \in M$ is defined to be the linear map

$$T_xf : T_xM \to T_{f(x)}N$$

constructed in the following way. For $v \in T_xM$, choose a curve $c :]-\epsilon, \epsilon[\to M$ with $c(0) = x$, and velocity vector $dc/dt\,|_{t=0} = v$. Then $T_xf \cdot v$ is the velocity vector at $t = 0$ of the curve $f \circ c : \mathbb{R} \to N$, that is,

$$T_xf \cdot v = \frac{d}{dt} f(c(t)) \bigg|_{t=0}.$$

The vector $T_x f \cdot v$ does not depend on the curve c but only on the vector v. If M and N are manifolds and $f : M \to N$ is of class C^{r+1}, then $Tf : TM \to TN$ is a mapping of class C^r. Note that

$$\left.\frac{dc}{dt}\right|_{t=0} = T_0 c \cdot 1.$$

Vector Fields and Flows. A ***vector field*** X on a manifold M is a map $X : M \to TM$ that assigns a vector $X(x)$ at the point $x \in M$; that is, $\tau_M \circ X =$ identity. An ***integral curve*** of X with initial condition x_0 at $t = 0$ is a (differentiable) map $c :]a, b[\to M$ such that $]a, b[$ is an open interval containing 0, $c(0) = x_0$ and

$$c'(t) = X(c(t))$$

for all $t \in]a, b[$. In formal presentations we usually suppress the domain of definition, even though this is technically important. The ***flow*** of X is the collection of maps

$$\varphi_t : M \to M$$

such that $t \mapsto \varphi_t(x)$ is the integral curve of X with initial condition x. Existence and uniqueness theorems from ordinary differential equations guarantee φ is smooth in x and t (where defined) if X is. From uniqueness, we get the ***flow property***

$$\varphi_{t+s} = \varphi_t \circ \varphi_s$$

along with the initial conditions $\varphi_0 =$ identity. The flow property generalizes the situation where $M = V$ is a *linear* space, $X(x) = Ax$ for a (bounded) *linear* operator A, and where

$$\varphi_t(x) = e^{tA} x$$

to the *nonlinear* case.

A ***time dependent vector field*** is a map $X : M \times \mathbb{R} \to TM$ such that $X(x, t) \in T_x M$ for each $x \in M$ and $t \in \mathbb{R}$. An ***integral curve*** of X is a curve $c(t)$ in M such that $c'(t) = X(c(t), t)$. Now the flow is the collection of maps

$$\varphi_{t,s} : M \to M$$

such that $t \mapsto \varphi_{t,s}(x)$ is the integral curve $c(t)$ such that $c(s) = x$. Again, the existence and uniqness theorem from ODE theory applies and, in particular, uniqueness gives the ***time dependent flow property***:

$$\varphi_{t,s} \circ \varphi_{s,r} = \varphi_{t,r}.$$

If X happens to be time independent, the two notions of flow are related by $\varphi_{t,s} = \varphi_{t-s}$.

Differentials and Covectors. If $f : M \to \mathbb{R}$ is a smooth function, we can differentiate it at any point $x \in M$ to obtain a map $T_x f : T_x M \to T_{f(x)}\mathbb{R}$. Identifying the tangent space of $\mathbb{R}$ at any point with itself (a process we usually do in any vector space), we get a linear map $\mathbf{d}f(x) : T_x M \to \mathbb{R}$. That is, $\mathbf{d}f(x) \in T_x^* M$, the dual of the vector space $T_x M$.

In coordinates, the ***directional derivatives*** defined by $\mathbf{d}f(x) \cdot v$, where $v \in T_x M$, are given by

$$\mathbf{d}f(x) \cdot v = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} v^i.$$

We will employ the ***summation convention*** and drop the summation sign when there are repeated indices. We also call $\mathbf{d}f$ the ***differential*** of f.

One can show that specifying the directional derivatives completely determines a vector, and so we can identify a basis of $T_x M$ using the operators $\partial/\partial x^i$. We write

$$(e_1, \ldots, e_n) = \left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n} \right)$$

for this basis so that $v = v^i \partial/\partial x^i$.

If we replace each vector space $T_x M$ with its dual $T_x^* M$, we obtain a new $2n$-manifold called the ***cotangent bundle*** and denoted T^*M. The dual basis to $\partial/\partial x^i$ is denoted dx^i. Thus, relative to a choice of local coordinates we get the basic formula

$$\mathbf{d}f(x) = \frac{\partial f}{\partial x^i} dx^i$$

for any smooth function $f : M \to \mathbb{R}$.

Exercises

$\diamond$ **Exercise 4.1-1.** Show that the two-sphere $S^2 \subset \mathbb{R}^3$ is a 2-manifold.

$\diamond$ **Exercise 4.1-2.** If $\varphi_t : S^2 \to S^2$ rotates points on S^2 about a fixed axis through an angle t, show that φ_t is the flow of a certain vector field on S^2.

$\diamond$ **Exercise 4.1-3.** Let $f : S^2 \to \mathbb{R}$ be defined by $f(x, y, z) = z$. Compute $\mathbf{d}f$ relative to spherical coordinates (θ, φ).

4.2 Differential Forms

We next review some of the basic definitions, properties, and operations on differential forms, without proofs (see Abraham, Marsden, and Ratiu [1988] and references therein). *The main idea of differential forms is to provide a generalization of the basic operations of vector calculus,* div*,* grad*, and* curl*, and the integral theorems of Green, Gauss, and Stokes to manifolds of arbitrary dimension.*

Basic Definitions. A ***2-form*** Ω on a manifold M is a function $\Omega(x) : T_xM \times T_xM \to \mathbb{R}$ that assigns to each point $x \in M$ a skew-symmetric bilinear form on the tangent space T_xM to M at x. More generally, a ***k-form*** α (sometimes called a ***differential form of degree*** k) on a manifold M is a function $\alpha(x) : T_xM \times \ldots \times T_xM$ (there are k factors) $\to \mathbb{R}$ that assigns to each point $x \in M$ a skew-symmetric k-multilinear map on the tangent space T_xM to M at x. Without the skew-symmetry assumption, α would be called a $(0,k)$-***tensor***. A map $\alpha : V \times \ldots \times V$ (there are k factors) $\to \mathbb{R}$ is ***multilinear*** when it is linear in each of its factors, that is,

$$\begin{aligned}\alpha(v_1, \ldots, av_j + bv_j', \ldots, v_k) \\ = a\alpha(v_1, \ldots, v_j, \ldots, v_k) + b\alpha(v_1, \ldots, v_j', \ldots, v_k)\end{aligned}$$

for all j with $1 \leq j \leq k$. A k-multilinear map $\alpha : V \times \ldots \times V \to \mathbb{R}$ is ***skew*** (or ***alternating***) when it changes sign whenever two of its arguments are interchanged, that is, for all $v_1, \ldots, v_k \in V$,

$$\alpha(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k) = -\alpha(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_k).$$

Let $x^1, \ldots, x^n$ denote coordinates on M, let

$$\{e_1, \ldots, e_n\} = \{\partial/\partial x^1, \ldots, \partial/\partial x^n\}$$

be the corresponding basis for T_xM, and let $\{e^1, \ldots, e^n\} = \{dx^1, \ldots, dx^n\}$ be the dual basis for T_x^*M. Then at each $x \in M$, we can write a 2-form as

$$\Omega_x(v, w) = \Omega_{ij}(x)v^i w^j, \quad \text{where} \quad \Omega_{ij}(x) = \Omega_x\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right),$$

and more generally a k-form can be written

$$\alpha_x(v_1, \ldots, v_k) = \alpha_{i_1 \ldots i_k}(x) v_1^{i_1} \ldots v_k^{i_k},$$

where there is a sum on $i_1, \ldots, i_k$ and where

$$\alpha_{i_1 \ldots i_k}(x) = \alpha_x\left(\frac{\partial}{\partial x^{i_1}}, \ldots, \frac{\partial}{\partial x^{i_k}}\right),$$

and where $v_i = v_i^j \partial/\partial x^j$, with a sum on j.

Tensor and Wedge Products. If α is a $(0,k)$-tensor on a manifold M, and β is a $(0,l)$-tensor, their ***tensor product*** $\alpha \otimes \beta$ is the $(0,k+l)$-tensor on M defined by

$$(\alpha \otimes \beta)_x(v_1, \ldots, v_{k+l}) = \alpha_x(v_1, \ldots, v_k)\beta_x(v_{k+1}, \ldots, v_{k+l}) \tag{4.2.1}$$

at each point $x \in M$.

If t is a $(0,p)$-tensor, define the ***alternation operator*** $\mathbf{A}$ acting on t by

$$\mathbf{A}(t)(v_1, \ldots, v_p) = \frac{1}{p!} \sum_{\pi \in S_p} \operatorname{sgn}(\pi) t(v_{\pi(1)}, \ldots, v_{\pi(p)}), \tag{4.2.2}$$

where $\operatorname{sgn}(\pi)$ is the ***sign*** of the permutation π:

$$\operatorname{sgn}(\pi) = \begin{cases} +1 & \text{if } \pi \text{ is even,} \\ -1 & \text{if } \pi \text{ is odd,} \end{cases} \tag{4.2.3}$$

and S_p is the group of all permutations of the numbers $1, 2, \ldots, p$. The operator $\mathbf{A}$ therefore skew-symmetrizes p-multilinear maps.

If α is a k-form and β is an l-form on M, their ***wedge product*** $\alpha \wedge \beta$ is the $(k+l)$-form on M defined by[1]

$$\alpha \wedge \beta = \frac{(k+l)!}{k!\, l!} \mathbf{A}(\alpha \otimes \beta). \tag{4.2.4}$$

For example, if α and β are one-forms,

$$(\alpha \wedge \beta)(v_1, v_2) = \alpha(v_1)\beta(v_2) - \alpha(v_2)\beta(v_1)$$

while if α is a 2-form and β is a 1-form,

$$(\alpha \wedge \beta)(v_1, v_2, v_3) = \alpha(v_1, v_2)\beta(v_3) + \alpha(v_3, v_1)\beta(v_2) + \alpha(v_2, v_3)\beta(v_1).$$

We state the following without proof:

Proposition 4.2.1. *The wedge product has the following properties:*

(i) $\alpha \wedge \beta$ *is* ***associative***: $\alpha \wedge (\beta \wedge \gamma) = (\alpha \wedge \beta) \wedge \gamma$.

(ii) $\alpha \wedge \beta$ *is* ***bilinear*** *in* α, β :

$$(a\alpha_1 + b\alpha_2) \wedge \beta = a(\alpha_1 \wedge \beta) + b(\alpha_2 \wedge \beta),$$
$$\alpha \wedge (c\beta_1 + d\beta_2) = c(\alpha \wedge \beta_1) + d(\alpha \wedge \beta_2).$$

[1] The numerical factor in (4.2.4) agrees with the convention of Abraham and Marsden [1978], Abraham, Marsden, and Ratiu [1988], and Spivak [1976], but *not* that of Arnold [1989], Guillemin and Pollack [1974], or Kobayashi and Nomizu [1963]; it is the Bourbaki [1971] convention.

(iii) *$\alpha \wedge \beta$ is **anticommutative**: $\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha$, where α is a k-form and β is an l-form.*

In terms of the dual basis dx^i, any k-form can be written locally as

$$\alpha = \alpha_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

where the sum is over all i_j satisfying $i_1 < \dots < i_k$.

Pull Back and Push Forward. Let $\varphi : M \to N$ be a C^∞ map from the manifold M to the manifold N and α be a k-form on N. Define the ***pull back*** $\varphi^* \alpha$ of α by φ to be the k-form on M given by

$$(\varphi^* \alpha)_x (v_1, \dots, v_k) = \alpha_{\varphi(x)} (T_x \varphi \cdot v_1, \dots, T_x \varphi \cdot v_k). \tag{4.2.5}$$

If φ is a diffeomorphism, the ***push forward*** φ_* is defined by $\varphi_* = (\varphi^{-1})^*$.

Here is another basic property.

Proposition 4.2.2. *The pull back of a wedge product is the wedge product of the pull backs:*

$$\varphi^* (\alpha \wedge \beta) = \varphi^* \alpha \wedge \varphi^* \beta. \tag{4.2.6}$$

Interior Products and Exterior Derivatives. Let α be a k-form on a manifold M and X a vector field. The ***interior product*** $\mathbf{i}_X \alpha$ (sometimes called the ***contraction*** of X and α, and written $X \lrcorner \, \alpha$) is defined by

$$(\mathbf{i}_X \alpha)_x (v_2, \dots, v_k) = \alpha_x (X(x), v_2, \dots, v_k). \tag{4.2.7}$$

Proposition 4.2.3. *Let α be a k-form and β an l-form on a manifold M. Then*

$$\mathbf{i}_X (\alpha \wedge \beta) = (\mathbf{i}_X \alpha) \wedge \beta + (-1)^k \alpha \wedge (\mathbf{i}_X \beta). \tag{4.2.8}$$

In the 'hook' notation, this reads

$$X \lrcorner \, (\alpha \wedge \beta) = (X \lrcorner \, \alpha) \wedge \beta + (-1)^k \alpha \wedge (X \lrcorner \, \beta).$$

The ***exterior derivative*** $\mathbf{d}\alpha$ of a k-form α on a manifold M is the $(k+1)$-form on M determined by the following proposition:

Proposition 4.2.4. *There is a unique mapping $\mathbf{d}$ from k-forms on M to $(k+1)$-forms on M such that:*

(i) *If α is a 0- form ($k = 0$), that is, $\alpha = f \in C^\infty(M)$, then $\mathbf{d}f$ is the one-form which is the differential of f.*

(ii) *$\mathbf{d}\alpha$ is **linear** in α, that is, for all real numbers c_1 and c_2,*

$$\mathbf{d}(c_1 \alpha_1 + c_2 \alpha_2) = c_1 \mathbf{d}\alpha_1 + c_2 \mathbf{d}\alpha_2.$$

(iii) $\mathbf{d}\alpha$ *satisfies the* ***product rule***, *that is,*

$$\mathbf{d}(\alpha \wedge \beta) = \mathbf{d}\alpha \wedge \beta + (-1)^k \alpha \wedge \mathbf{d}\beta,$$

where α is a k-form and, β is an l-form.

(iv) $\mathbf{d}^2 = 0$, *that is,* $\mathbf{d}(\mathbf{d}\alpha) = 0$ *for any k-form α.*

(v) $\mathbf{d}$ *is a* ***local operator***, *that is,* $\mathbf{d}\alpha(x)$ *only depends on α restricted to any open neighborhood of x; in fact, if U is open in M, then*

$$\mathbf{d}(\alpha|U) = (\mathbf{d}\alpha)|U.$$

If α is a k-form given in coordinates by

$$\alpha = \alpha_{i_1 \ldots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k} \quad (\text{sum on } i_1 < \cdots < i_k),$$

then the coordinate expression for the exterior derivative is

$$\mathbf{d}\alpha = \frac{\partial \alpha_{i_1 \ldots i_k}}{\partial x^j} dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} \qquad (\text{sum on all } j \text{ and } i_1 < \cdots < i_k) \tag{4.2.9}$$

Formula (4.2.9) can be taken as the definition of the exterior derivative, provided one shows that (4.2.9) has the above-described properties and, correspondingly, is independent of the choice of coordinates.

Next is a useful proposition that, in essence, rests on the chain rule:

Proposition 4.2.5. *Exterior differentiation commutes with pull back, that is,*

$$\mathbf{d}(\varphi^* \alpha) = \varphi^*(\mathbf{d}\alpha), \tag{4.2.10}$$

where α is a k-form on a manifold N and φ is a smooth map from a manifold M to N.

A k-form α is called ***closed*** if $\mathbf{d}\alpha = 0$ and *exact* if there is a $(k-1)$-form β such that $\alpha = \mathbf{d}\beta$. By Proposition **4.2.4iv** every exact form is closed. Exercise **4.4-2** gives an example of a closed nonexact one-form.

Proposition 4.2.6 (Poincaré Lemma). *A closed form is locally exact, that is, if $\mathbf{d}\alpha = 0$ there is a neighborhood about each point on which $\alpha = \mathbf{d}\beta$.*

See Exercise **4.2-5** for the proof.

Vector Calculus. The table below entitled "Vector calculus and differential forms" summarizes how forms are related to the usual operations of vector calculus. We now elaborate on a few items in this table. In item 4, note that

$$\mathbf{d}f = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz = (\operatorname{grad} f)^\flat = (\nabla f)^\flat$$

which is equivalent to $\nabla f = (\mathbf{d}f)^\sharp$.

The Hodge star operator on $\mathbb{R}^3$ maps k-forms to $(3-k)$-forms and is uniquely determined by linearity and the properties in item 2. (This operator can be defined on general Riemannian manifolds; see Abraham, Marsden, and Ratiu [1988].)

In item 5, if we let $F = F_1\mathbf{e}_1 + F_2\mathbf{e}_2 + F_3\mathbf{e}_3$, so $F^\flat = F_1\,dx + F_2\,dy + F_3\,dz$, then,

$$\begin{aligned}
\mathbf{d}(F^\flat) &= \mathbf{d}F_1 \wedge dx + F_1\mathbf{d}(dx) + \mathbf{d}F_2 \wedge dy + F_2\mathbf{d}(dy) \\
&\quad + \mathbf{d}F_3 \wedge dz + F_3\mathbf{d}(dz) \\
&= \left(\frac{\partial F_1}{\partial x}dx + \frac{\partial F_1}{\partial y}dy + \frac{\partial F_1}{\partial z}dz\right) \wedge dx \\
&\quad + \left(\frac{\partial F_2}{\partial x}dx + \frac{\partial F_2}{\partial y}dy + \frac{\partial F_2}{\partial z}dz\right) \wedge dy \\
&\quad + \left(\frac{\partial F_3}{\partial x}dx + \frac{\partial F_3}{\partial y}dy + \frac{\partial F_3}{\partial z}dz\right) \wedge dz \\
&= -\frac{\partial F_1}{\partial y}dx \wedge dy + \frac{\partial F_1}{\partial z}dz \wedge dx + \frac{\partial F_2}{\partial x}dx \wedge dy - \frac{\partial F_2}{\partial z}dy \wedge dz \\
&\quad - \frac{\partial F_3}{\partial x}dz \wedge dx + \frac{\partial F_3}{\partial y}dy \wedge dz \\
&= \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx \wedge dy + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) dz \wedge dx \\
&\quad + \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) dy \wedge dz.
\end{aligned}$$

Hence, using item 2,

$$\begin{aligned}
*(\mathbf{d}(F^\flat)) &= \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dz + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) dy + \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) dx, \\
(*(\mathbf{d}(F^\flat)))^\sharp &= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) \mathbf{e}_1 + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) \mathbf{e}_2 + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) \mathbf{e}_3 \\
&= \operatorname{curl} F = \nabla \times F.
\end{aligned}$$

With reference to item 6, let $F = F_1\mathbf{e}_1 + F_2\mathbf{e}_2 + F_3\mathbf{e}_3$, so

$$F^\flat = F_1\,dx + F_2\,dy + F_3\,dz.$$

Thus $*(F^\flat) = F_1\, dy \wedge dz + F_2(-dx \wedge dz) + F_3\, dx \wedge dy$, and so

$$
\begin{aligned}
\mathbf{d}(*(F^\flat)) &= \mathbf{d}F_1 \wedge dy \wedge dz - \mathbf{d}F_2 \wedge dx \wedge dz + \mathbf{d}F_3 \wedge dx \wedge dy \\
&= \left(\frac{\partial F_1}{\partial x}dx + \frac{\partial F_1}{\partial y}dy + \frac{\partial F_1}{\partial z}dz\right) \wedge dy \wedge dz \\
&\quad - \left(\frac{\partial F_2}{\partial x}dx + \frac{\partial F_2}{\partial y}dy + \frac{\partial F_2}{\partial z}dz\right) \wedge dx \wedge dz \\
&\quad + \left(\frac{\partial F_3}{\partial x}dx + \frac{\partial F_3}{\partial y}dy + \frac{\partial F_3}{\partial z}dz\right) \wedge dx \wedge dy \\
&= \frac{\partial F_1}{\partial x}dx \wedge dy \wedge dz + \frac{\partial F_2}{\partial y}dx \wedge dy \wedge dz + \frac{\partial F_3}{\partial z}dx \wedge dy \wedge dz \\
&= \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx \wedge dy \wedge dz = (\operatorname{div} F)\, dx \wedge dy \wedge dz.
\end{aligned}
$$

Therefore, $*(\mathbf{d}(*(F^\flat))) = \operatorname{div} F = \nabla \cdot F$.

The definition and properties of vector-valued forms are direct extensions of these for usual forms on vector spaces and manifolds. One can think of a vector-valued form as an array of usual forms (see Abraham, Marsden, and Ratiu [1988]).

Vector Calculus and Differential Forms

1. **Sharp and Flat** (Using standard coordinates in $\mathbb{R}^3$)

   (a) $v^\flat = v^1\, dx + v^2\, dy + v^3\, dz =$
   one-form corresponding to the vector
   $v = v^1\mathbf{e}_1 + v^2\mathbf{e}_2 + v^3\mathbf{e}_3$.

   (b) $\alpha^\sharp = \alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 + \alpha_3\mathbf{e}_3 =$
   vector corresponding to the one-form
   $\alpha = \alpha_1\, dx + \alpha_2\, dy + \alpha_3\, dz$.

2. **Hodge Star Operator**

   (a) $*1 = dx \wedge dy \wedge dz$.

   (b) $*dx = dy \wedge dz$, $*dy = -dx \wedge dz$, $*dz = dx \wedge dy$,
   $*(dy \wedge dz) = dx$, $*(dx \wedge dz) = -dy$, $*(dx \wedge dy) = dz$.

   (c) $*(dx \wedge dy \wedge dz) = 1$.

3. **Cross Product and Dot Product**

   (a) $v \times w = [*(v^\flat \wedge w^\flat)]^\sharp$.

   (b) $(v \cdot w)dx \wedge dy \wedge dz = v^\flat \wedge *(w^\flat)$.

4. Gradient $\nabla f = \operatorname{grad} f = (\mathbf{d}f)^\sharp$.

5. Curl $\nabla \times F = \operatorname{curl} F = [*(\mathbf{d}F^\flat)]^\sharp$.

6. Divergence $\nabla \cdot F = \operatorname{div}\, F = *\mathbf{d}(*F^\flat)$.

Exercises

◇ **Exercise 4.2-1.** Let $\varphi : \mathbb{R}^3 \to \mathbb{R}^2$ be given by $\varphi(x, y, z) = (x + z, xy)$. For $\alpha = e^v\, du + u\, dv \in \Omega^1(\mathbb{R}^2)$ and $\beta = u\, du \wedge dv$ compute $\alpha \wedge \beta, \varphi^*\alpha, \varphi^*\beta$, and $\varphi^*\alpha \wedge \varphi^*\beta$.

◇ **Exercise 4.2-2.** Given

$$\alpha = y^2\, dx \wedge dz + \sin(xy)\, dx \wedge dy + e^x\, dy \wedge dz \in \Omega^2(\mathbb{R}^3)$$

and

$$X = 3\partial/\partial x + \cos z \partial/\partial y - x^2 \partial/\partial z \in \mathfrak{X}(\mathbb{R}^3),$$

compute $\mathbf{d}\alpha$ and $\mathbf{i}_X\alpha$.

◇ **Exercise 4.2-3.**

(a) Denote by $\Lambda^k(\mathbb{R}^n)$ the vector space of all skew-symmetric k-linear maps on $\mathbb{R}^n$. Prove that this space has dimension $n!/k!\,(n-k)!$ by showing that a basis is given by $\{e^{i_1} \wedge \cdots \wedge e^{i_k} \mid i_1 < \ldots < i_k\}$, where $\{e_1, \ldots, e_n\}$ is a basis of $\mathbb{R}^n$ and $\{e^1, \ldots, e^n\}$ is its dual basis, that is, $e^i(e_j) = \delta^i_j$.

(b) If $\mu \in \Lambda^n(\mathbb{R}^n)$ is nonzero, prove that the map $v \in \mathbb{R}^n \mapsto \mathbf{i}_v\mu \in \Lambda^{n-1}(\mathbb{R}^n)$ is an isomorphism.

(c) If M is a smooth n-manifold and $\mu \in \Omega^n(M)$ is nowhere vanishing (in which case it is called a volume form), show that the map $X \in \mathfrak{X}(M) \mapsto \mathbf{i}_X\mu \in \Omega^{n-1}(M)$ is a module isomorphism over $\mathcal{F}(M)$.

◇ **Exercise 4.2-4.** Let $\alpha = \alpha_i\, dx^i$ be a closed one-form in a ball around the origin in $\mathbb{R}^n$. Show that $\alpha = \mathbf{d}f$ for

$$f(x^1, \ldots, x^n) = \int_0^1 \alpha_j(tx^1, \ldots, tx^n) x^j\, dt.$$

◇ **Exercise 4.2-5.**

(a) Let U be an open ball around the origin in $\mathbb{R}^n$ and $\alpha \in \Omega^k(U)$ a closed form. Verify that $\alpha = \mathbf{d}\beta$, where

$$\beta(x^1, \dots, x^n) = \left(\int_0^1 t^{k-1} \alpha_{j i_1 \dots i_{k-1}}(tx^1, \dots, tx^n) x^j \, dt \right) dx^{i_1} \wedge \dots \wedge dx^{i_{k-1}},$$

and where the sum is over $i_1 < \dots < i_{k-1}$. Here, $\alpha = \alpha_{j_1 \dots j_k} \, dx^{j_1} \wedge \dots \wedge dx^{j_k}$, where $j_1 < \dots < j_k$ and where α is extended to be skew-symmetric in its lower indices.

(b) Deduce the Poincaré lemma from **(a)**.

◇ **Exercise 4.2-6.** (Construction of a homotopy operator for a retraction.) Let M be a smooth manifold and $N \subset M$ a smooth submanifold. A family of smooth maps $r_t : M \to M$, $t \in [0,1]$, is called a ***retraction of*** M ***onto*** N, if $r_t|N =$ identity on N for all $t \in [0,1]$, $r_1 =$ identity on M, r_t is a diffeomorphism of M with $r_t(M)$ for every $t \neq 0$, and $r_0(M) = N$. Let X_t be the time dependent vector field generated by r_t, $t \neq 0$. Show that the operator $\mathbf{H} : \Omega^k(M) \to \Omega^{k-1}(M)$ defined by

$$\mathbf{H} = \int_0^1 (r_t^* \mathbf{i}_{X_t} \alpha) \, dt$$

satisfies

$$\alpha - (r_0^* \alpha) = \mathbf{dH}\alpha + \mathbf{Hd}\alpha.$$

Deduce the ***relative Poincaré lemma*** from this formula: if $\alpha \in \Omega^k(M)$ is closed and $\alpha|N = 0$, then there is a neigborhood U of N such that $\alpha|U = \mathbf{d}\beta$, for some $\beta \in \Omega^{k-1}(U)$ and $\beta|N = 0$. (Hint: Use the existence of a tubular neigborhood of N in M.).

4.3 The Lie Derivative

Lie Derivative Theorem. The *dynamic definition* of the Lie derivative is as follows. Let α be a k-form and let X be a vector field with flow φ_t. The ***Lie derivative*** of α along X is given by

$$\pounds_X \alpha = \lim_{t \to 0} \frac{1}{t} [(\varphi_t^* \alpha) - \alpha] = \left. \frac{d}{dt} \varphi_t^* \alpha \right|_{t=0}. \tag{4.3.1}$$

This definition together with properties of pull-backs yields the following.

Theorem 4.3.1 (Lie Derivative Theorem).

$$\frac{d}{dt} \varphi_t^* \alpha = \varphi_t^* \pounds_X \alpha. \tag{4.3.2}$$

This formula holds also for *time-dependent* vector fields in the sense that

$$\frac{d}{dt}\varphi_{t,s}^*\alpha = \varphi_{t,s}^*\pounds_X\alpha$$

and in $\Lambda_X^\alpha\alpha$, the vector field is evaluated at time t.

If f is a real-valued function on a manifold M and X is a vector field on M, the ***Lie derivative of*** f ***along*** X is the ***directional derivative***

$$\pounds_X f = X[f] := \mathbf{d}f \cdot X. \tag{4.3.3}$$

If M is finite-dimensional,

$$\pounds_X f = X^i \frac{\partial f}{\partial x^i}. \tag{4.3.4}$$

For this reason one often writes

$$X = X^i \frac{\partial}{\partial x^i}.$$

If Y is a vector field on a manifold N and $\varphi : M \to N$ is a diffeomorphism, the ***pull back*** φ^*Y is a vector field on M defined by

$$(\varphi^*Y)(x) = T_x\varphi^{-1} \circ Y \circ \varphi(x). \tag{4.3.5}$$

Two vector fields X on M and Y on N are said to be φ-***related*** if

$$T\varphi \circ X = Y \circ \varphi. \tag{4.3.6}$$

Clearly, if $\varphi : M \to N$ is a diffeomorphism and Y is a vector field on N, φ^*Y and Y are φ-related. For a diffeomorphism φ, the ***push forward*** is defined, as for forms, by $\varphi_* = (\varphi^{-1})^*$.

Jacobi–Lie Brackets. If M is finite dimensional and C^∞ then the set of vector fields on M coincides with the set of derivations on $\mathcal{F}(M)$. The same result is true for C^k manifolds and vector fields if $k \geq 2$. This property is false for infinite-dimensional manifolds; see Abraham, Marsden, Ratiu [1988]. If M is C^∞ and smooth, then the derivation $f \mapsto X[Y[f]] - Y[X[f]]$, where $X[f] = \mathbf{d}f \cdot X$, determines a unique vector field denoted by $[X, Y]$ and called the ***Jacobi–Lie bracket*** of X and Y. Defining $\pounds_X Y = [X, Y]$ gives the ***Lie derivative*** of Y along X. Then the Lie derivative theorem (4.3.2) holds with α replaced by Y and the pull back operation given by (4.3.5).

If M is infinite-dimensional, then one defines the Lie derivative of Y along X by

$$\left.\frac{d}{dt}\right|_{t=0} \varphi_t^*Y = \pounds_X Y, \tag{4.3.7}$$

where φ_t is the flow of X. Then formula (4.3.2) with α replaced by Y holds and the action of the vector field $\pounds_X Y$ on a function f is given by $X[Y[f]] - Y[X[f]]$ which is denoted, as in the finite-dimensional case, $[X, Y][f]$. As before $[X, Y] = \pounds_X Y$ is also called the Jacobi–Lie bracket of vector fields.

If M is finite-dimensional,

$$(\pounds_X Y)^j = X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i} = (X \cdot \nabla) Y^i - (Y \cdot \nabla) X^j, \tag{4.3.8}$$

and in general, where we identify X, Y with their local representatives

$$[X, Y] = \mathbf{D}Y \cdot X - \mathbf{D}X \cdot Y. \tag{4.3.9}$$

The formula for $[X, Y] = \pounds_X Y$ can be remembered by writing

$$\left[X^i \frac{\partial}{\partial x^i}, Y^j \frac{\partial}{\partial x^j} \right] = X^i \frac{\partial Y^j}{\partial x^i} \frac{\partial}{\partial x^j} - Y^j \frac{\partial X^i}{\partial x^j} \frac{\partial}{\partial x^i}.$$

Algebraic Definiton of the Lie Derivative. The *algebraic approach* to the Lie derivative on forms or tensors proceeds as follows. Extend the definition of the Lie derivative from functions and vector fields to differential forms, by requiring that the Lie derivative is a derivation; for example, for one-forms α, write

$$\pounds_X \langle \alpha, Y \rangle = \langle \pounds_X \alpha, Y \rangle + \langle \alpha, \pounds_X Y \rangle \,, \tag{4.3.10}$$

where X, Y are vector fields and $\langle \alpha, Y \rangle = \alpha(Y)$. More generally,

$$\pounds_X (\alpha(Y_1, \ldots, Y_k)) = (\pounds_X \alpha)(Y_1, \ldots, Y_k) + \sum_{i=1}^{k} \alpha(Y_1, \ldots, \pounds_X Y_i, \ldots, Y_k), \tag{4.3.11}$$

where $X, Y_1, \ldots, Y_k$ are vector fields and α is a k-form.

Proposition 4.3.2. *The dynamic and algebraic definitions of the Lie derivative of a differential k-form are equivalent.*

Cartan's Magic Formula. A very important formula for the Lie derivative is given by the following.

Theorem 4.3.3. *For X a vector field and α a k-form on a manifold M, we have*

$$\pounds_X \alpha = \mathbf{d} \mathbf{i}_X \alpha + \mathbf{i}_X \mathbf{d} \alpha. \tag{4.3.12}$$

This is proved by a lengthy but straightforward calculation.

Another property of the Lie derivative is the following: if $\varphi : M \to N$ is a diffeomorphism,

$$\varphi^* \pounds_Y \beta = \pounds_{\varphi^* Y} \varphi^* \beta$$

for $Y \in \mathfrak{X}(N), \beta \in \Omega^k(M)$. More generally, if $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ are ψ related, that is, $T\psi \circ X = Y \circ \psi$ for $\psi : M \to N$ a smooth map, then $\pounds_X \psi^* \beta = \psi^* \pounds_Y \beta$ for all $\beta \in \Omega^k(N)$.

Volume Forms and Divergence. An n-manifold M is said to be ***orientable*** if there is a nowhere vanishing n-form μ on it; μ is called a ***volume form*** and it is a basis of $\Omega^n(M)$ over $\mathcal{F}(M)$. Two volume forms μ_1 and μ_2 on M are said to define the same ***orientation*** if there is an $f \in \mathcal{F}(M)$, with $f > 0$ and such that $\mu_2 = f\mu_1$. Connected orientable manifolds admit precisely two orientations. A basis $\{v_1, \dots v_n\}$ of $T_m M$ is said to be ***positively oriented*** relative to the volume form μ on M if $\mu(m)(v_1, \dots, v_n) > 0$. Note that the volume forms defining the same orientation form a convex cone in $\Omega^n(M)$, that is, if $a > 0$ and μ is a volume form, then $a\mu$ is again a volume form and if $t \in [0, 1]$ and μ_1, μ_2 are volume forms, then $t\mu_1 + (1-t)\mu_2$ is again a volume form. The first property is obvious. To prove the second, let $m \in M$ and let $\{v_1, \dots v_n\}$ be a positively oriented basis of $T_m M$ relative to the orientation defined by μ_1, or equivalently (by hypothesis) by μ_2. Then $\mu_1(m)(v_1, \dots, v_n) > 0$, $\mu_2(m)(v_1, \dots, v_n) > 0$ so that their convex combination is again strictly positive.

If $\mu \in \Omega^n(M)$ is a volume form, since $\pounds_X \mu \in \Omega^n(M)$ there is a function, called the ***divergence*** of X relative to μ and denoted $\operatorname{div}_\mu(X)$ or simply $\operatorname{div}(X)$, such that

$$\pounds_X \mu = \operatorname{div}_\mu(X)\mu. \tag{4.3.13}$$

From the dynamic approach to Lie derivatives it follows that $\operatorname{div}_\mu(X) = 0$ iff $F_t^* \mu = \mu$, where F_t is the flow of X. This condition says that F_t is ***volume preserving***. If $\varphi : M \to M$, since $\varphi^* \mu \in \Omega^n(M)$ there is a function, called the ***Jacobian*** of φ and denoted $J_\mu(\varphi)$ or simply $J(\varphi)$, such that

$$\varphi^* \mu = J_\mu(\varphi)\mu. \tag{4.3.14}$$

Thus, *φ is volume preserving iff $J_\mu(\varphi) = 1$. From the inverse function theorem, we see that φ is a local diffeomorphism iff $J_\mu(\varphi) \neq 0$ on M.*

There are a number of valuable identities relating the Lie derivative, the exterior derivative and the interior product. For example, if Θ is a one form and X and Y are vector fields, identity 6 in the following table gives

$$\mathbf{d}\Theta(X, Y) = X[\Theta(Y)] - Y[\Theta(X)] - \Theta([X, Y]). \tag{4.3.15}$$

Exercises

◇ **Exercise 4.3-1.** Let M be an n-manifold, $\mu \in \Omega^n(M)$ a volume form, $X, Y \in \mathfrak{X}(M)$, and $f, g : M \to \mathbb{R}$ smooth functions such that $f(m) \neq 0$ for all m. Prove the following identities:

(a) $\operatorname{div}_{f\mu}(X) = \operatorname{div}_\mu(X) + X[f]/f$;

(b) $\operatorname{div}_\mu(gX) = g \operatorname{div}_\mu(X) + X[g]$; and

(c) $\operatorname{div}_\mu([X, Y]) = X[\operatorname{div}_\mu(Y)] - Y[\operatorname{div}_\mu(X)]$.

◇ **Exercise 4.3-2.** Show that the partial differential equation

$$\frac{\partial f}{\partial t} = \sum_{i=1}^{n} X^i(x^1, \ldots, x^n) \frac{\partial f}{\partial x^i}$$

with initial condition $f(x, 0) = g(x)$ has the solution $f(x, t) = g(F_t(x))$, where F_t is the flow of the vector field $(X^1, \ldots, X^n)$ in $\mathbb{R}^n$ whose flow is assumed to exist for all time. Show that the solution is *unique*. Generalize this exercise to the equation

$$\frac{\partial f}{\partial t} = X[f]$$

for X a vector field on a manifold M.

◇ **Exercise 4.3-3.** Show that if M and N are orientable manifolds, so is $M \times N$.

4.4 Stokes' Theorem

The basic idea of the definition of the integral of an n-form μ on an oriented n-manifold M is to pick a covering by coordinate charts and to sum up the ordinary integrals of $f(x^1, \ldots, x^n)\, dx^1 \cdots dx^n$, where

$$\mu = f(x^1, \ldots, x^n)\, dx^1 \wedge \cdots \wedge dx^n$$

is the local representative of μ, being careful not to count overlaps twice. The change of variables formula guarantees that the result, denoted by $\int_M \mu$, is well defined.

If one has an oriented manifold with boundary, then the boundary, ∂M, inherits a compatible orientation. This proceeds in a way that generalizes the relation between the orientation of a surface and its boundary in the classical Stokes' theorem in $\mathbb{R}^3$.

Theorem 4.4.1. (Stokes' Theorem) *Suppose that M is a compact, oriented k-dimensional manifold with boundary ∂M. Let α be a smooth $(k-1)$-form on M. Then*

$$\int_M \mathbf{d}\alpha = \int_{\partial M} \alpha. \tag{4.4.1}$$

Special cases of Stokes' theorem are as follows:

The Integral Theorems of Calculus. Stokes' theorem generalizes and synthesizes the classical theorems:

(a) Fundamental Theorem of Calculus.

$$\int_b^a f'(x)\, dx = f(b) - f(a). \tag{4.4.2}$$

(b) Green's Theorem. For a region $\Omega \subset \mathbb{R}^2$:

$$\iint_\Omega \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy = \int_{\partial\Omega} P\, dx + Q\, dy. \tag{4.4.3}$$

(c) Divergence Theorem. For a region $\Omega \subset \mathbb{R}^3$:

$$\iiint_\Omega \operatorname{div} \mathbf{F}\, dV = \iint_{\partial\Omega} \mathbf{F} \cdot n\, dA. \tag{4.4.4}$$

(d) Classical Stokes' Theorem. For a surface $S \subset \mathbb{R}^3$:

$$\begin{aligned} &\iint_S \left\{ \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) dy \wedge dz \right. \\ &\qquad \left. + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) dz \wedge dx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \wedge dy \right\} \\ &= \iint_S \mathbf{n} \cdot \operatorname{curl} \mathbf{F}\, dA \\ &= \int_{\partial S} P\, dx + Q\, dy + R\, dz, \end{aligned} \tag{4.4.5}$$

where $\mathbf{F} = (P, Q, R)$.

Notice that the Poincaré lemma generalizes the vector calculus theorems in $\mathbb{R}^3$ saying that if $\operatorname{curl} \mathbf{F} = 0$, then $\mathbf{F} = \nabla f$ and if $\operatorname{div} \mathbf{F} = 0$, then $\mathbf{F} = \nabla \times \mathbf{G}$. Recall that it states: *If α is closed, then locally α is exact; that is, if $\mathbf{d}\alpha = 0$, then locally $\alpha = \mathbf{d}\beta$ for some β.*

Cohomology. The failure of closed forms to be globally exact leads to the study of a very important topological invariant of M, the ***de Rham cohomology***. The kth de Rham cohomology group, denoted $H^k(M)$ is defined by

$$H^k(M) := \frac{\ker(\mathbf{d} : \Omega^k(M) \to \Omega^{k+1}(M))}{\operatorname{range}(\mathbf{d} : \Omega^{k-1}(M) \to \Omega^k(M))}.$$

The de Rham theorem states that these abelian groups are isomorphic to the so-called singular cohomology groups of M defined in algebraic topology in terms of simplexes and that depend only on the topological structure of M and not on its differentiable structure. The isomorphism is provided by integration and the fact that the integration map drops to the preceding quotient is guaranteed by Stokes' theorem. A useful particular case of this theorem is the following: if M is an orientable compact boundaryless n-manifold, then $\int_M \mu = 0$ if and only if the n-form μ is exact. This statement is equivalent to $H^n(M) = \mathbb{R}$.

Change of Variables. Another basic result in integration theory is the global change of variables formula.

Theorem 4.4.2 (Change of Variables). *Let M and N be oriented n-manifolds and let $F : M \to N$ be an orientation-preserving diffeomorphism. If α is an n-form on N (with, say, compact support), then*

$$\int_M F^*\alpha = \int_N \alpha.$$

Frobenius' Theorem. We also mention a basic result called ***Frobenius' theorem***. If $E \subset TM$ is a vector subbundle, it is said to be ***involutive*** if for any two vector fields X, Y on M with values in E, $[X, Y]$ is also a vector field with values in E. The subbundle E is said to be ***integrable*** if for each point $m \in M$ there is a local submanifold of M containing m such that its tangent bundle equals E restricted to this submanifold. If E is integrable, the local integral manifolds can be extended to get, through each $m \in M$, a maximal integral manifold, which is an immersed submanifold of M. The collection of all maximal integral manifolds through all points of M forms a foliation.

The Frobenius theorem states that the involutivity of E is equivalent to the integrability of E, which in turn is equivalent to the existence of a foliation on M whose tangent bundle equals E.

Identities for Vector Fields and Forms

1. Vector fields on M with the bracket $[X, Y]$ form a ***Lie algebra***; that is, $[X, Y]$ is real bilinear, skew-symmetric, and ***Jacobi's identity***

holds:

$$[[X, Y], Z] + [[Z, X], Y] + [[Y, Z], X] = 0.$$

Locally,

$$[X, Y] = \mathbf{D}Y \cdot X - \mathbf{D}X \cdot Y = (X \cdot \nabla)Y - (Y \cdot \nabla)X$$

and on functions,

$$[X, Y][f] = X[Y[f]] - Y[X[f]].$$

2. For diffeomorphisms φ and ψ,

$$\varphi_*[X, Y] = [\varphi_* X, \varphi_* Y] \quad \text{and} \quad (\varphi \circ \psi)_* X = \varphi_* \psi_* X.$$

3. The forms on a manifold comprise a real associative algebra with $\wedge$ as multiplication. Furthermore, $\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha$ for k and l-forms α and β, respectively.

4. For maps φ and ψ,

$$\varphi^*(\alpha \wedge \beta) = \varphi^*\alpha \wedge \varphi^*\beta \quad \text{and} \quad (\varphi \circ \psi)^*\alpha = \psi^*\varphi^*\alpha.$$

5. $\mathbf{d}$ is a real linear map on forms, $\mathbf{dd}\alpha = 0$, and

$$\mathbf{d}(\alpha \wedge \beta) = \mathbf{d}\alpha \wedge \beta + (-1)^k \alpha \wedge \mathbf{d}\beta$$

for α a k-form.

6. For α a k-form and $X_0, \ldots, X_k$ vector fields,

$$\begin{aligned}(\mathbf{d}\alpha)(X_0, \ldots, X_k) &= \sum_{i=0}^{k} (-1)^i X_i[\alpha(X_0, \ldots, \hat{X}_i, \ldots, X_k)] \\ &\quad + \sum_{0 \le i < j \le k} (-1)^{i+j} \alpha([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_k)\end{aligned}$$

where $\hat{X}_i$ means that X_i is omitted. Locally,

$$\mathbf{d}\alpha(x)(v_0, \ldots, v_k) = \sum_{i=0}^{k} (-1)^i \mathbf{D}\alpha(x) \cdot v_i(v_0, \ldots, \hat{v}_i, \ldots, v_k).$$

7. For a map φ, $\qquad \varphi^*\mathbf{d}\alpha = \mathbf{d}\varphi^*\alpha.$

8. Poincaré Lemma. If $\mathbf{d}\alpha = 0$, then α is locally exact; that is, there is a neighborhood U about each point on which $\alpha = \mathbf{d}\beta$. The same result holds globally on a contractible manifold.

9. $\mathbf{i}_X\alpha$ is real bilinear in X, α and for $h : M \to \mathbb{R}$,

$$\mathbf{i}_{hX}\alpha = h\mathbf{i}_X\alpha = \mathbf{i}_X h\alpha.$$

Also, $\mathbf{i}_X\mathbf{i}_X\alpha = 0$ and

$$\mathbf{i}_X(\alpha \wedge \beta) = \mathbf{i}_X\alpha \wedge \beta + (-1)^k \alpha \wedge \mathbf{i}_X\beta$$

for α a k-form.

10. For a diffeomorphism φ,

$$\varphi^*(\mathbf{i}_X\alpha) = \mathbf{i}_{\varphi^* X}(\varphi^*\alpha);$$

if $f : M \to N$ is a mapping and Y is f-related to X, that is,

$$Tf \circ X = Y \circ f,$$

then

$$\mathbf{i}_X f^*\alpha = f^*\mathbf{i}_Y\alpha.$$

11. $\pounds_X\alpha$ is real bilinear in X, α and

$$\pounds_X(\alpha \wedge \beta) = \pounds_X\alpha \wedge \beta + \alpha \wedge \pounds_X\beta.$$

12. Cartan's Magic Formula:

$$\pounds_X\alpha = \mathbf{d}\mathbf{i}_X\alpha + \mathbf{i}_X\mathbf{d}\alpha.$$

13. For a diffeomorphism φ,

$$\varphi^*\pounds_X\alpha = \pounds_{\varphi^* X}\varphi^*\alpha;$$

if $f : M \to N$ is a mapping and Y is f-related to X, then

$$\pounds_Y f^*\alpha = f^*\pounds_X\alpha.$$

14.
$$\begin{aligned}(\pounds_X\alpha)(X_1, \dots, X_k) = X[\alpha(X_1, \dots, X_k)]\\ - \sum_{i=0}^{k} \alpha(X_1, \dots, [X, X_i], \dots, X_k).\end{aligned}$$

Locally,

$$\begin{aligned}(\pounds_X\alpha)(x) \cdot (v_1, \dots, v_k) = (\mathbf{D}\alpha_x \cdot X(x))(v_1, \dots, v_k)\\ + \sum_{i=0}^{k} \alpha_x(v_1, \dots, \mathbf{D}X_x \cdot v_i, \dots, v_k).\end{aligned}$$

15. The following identities hold:

(a) $\pounds_{fX}\alpha = f\pounds_X\alpha + \mathbf{d}f \wedge \mathbf{i}_X\alpha$;

(b) $\pounds_{[X,Y]}\alpha = \pounds_X\pounds_Y\alpha - \pounds_Y\pounds_X\alpha$;

(c) $\mathbf{i}_{[X,Y]}\alpha = \pounds_X\mathbf{i}_Y\alpha - \mathbf{i}_Y\pounds_X\alpha$;

(d) $\pounds_X\mathbf{d}\alpha = \mathbf{d}\pounds_X\alpha$; and

(e) $\pounds_X\mathbf{i}_X\alpha = \mathbf{i}_X\pounds_X\alpha$.

(f) $\pounds_X(\alpha \wedge \beta) = \pounds_X\alpha \wedge \beta + \alpha \wedge \pounds_X\beta$

16. If M is a finite-dimensional manifold, $X = X^l\partial/\partial x^l$, and

$$\alpha = \alpha_{i_1 \dots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k},$$

where $i_1 < \cdots < i_k$, then the following formulas hold:

$$\begin{aligned}
\mathbf{d}\alpha &= \left(\frac{\partial \alpha_{i_1 \dots i_k}}{\partial x^l}\right) dx^l \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k},\\
\mathbf{i}_X\alpha &= X^l \alpha_{l i_2 \dots i_k} dx^{i_2} \wedge \cdots \wedge dx^{i_k},\\
\pounds_X\alpha &= X^l \left(\frac{\partial \alpha_{i_1 \dots i_k}}{\partial x^l}\right) dx^{i_1} \wedge \cdots \wedge dx^{i_k}\\
&\qquad + \ \alpha_{l i_2 \dots i_k} \left(\frac{\partial X^l}{\partial x^{i_1}}\right) dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_k} + \ldots .
\end{aligned}$$

Exercises

◇ **Exercise 4.4-1.** Let Ω be a closed bounded region in $\mathbb{R}^2$. Use Green's theorem to show that the area of Ω equals the line integral

$$\frac{1}{2}\int_{\partial\Omega} (x\,dy - y\,dx).$$

◇ **Exercise 4.4-2.** On $\mathbb{R}^2\backslash\{(0,0)\}$ consider the one-form

$$\alpha = (x\,dy - y\,dx)/(x^2 + y^2).$$

(a) Show that this form is closed.

(b) Using the angle θ as a variable on S^1, compute $i^*\alpha$, where $i : S^1 \to \mathbb{R}^2$ is the standard embedding.

(c) Show that α is not exact.

◇ **Exercise 4.4-3.** **The magnetic monopole** Let $\mathbf{B} = g\mathbf{r}/r^3$ be a vector field on Euclidean three-space minus the origin where $r = \|\mathbf{r}\|$. Show that $\mathbf{B}$ cannot be written as the curl of something.