

# 10

## Poisson Manifolds

The dual  $\mathfrak{g}^*$  of a Lie algebra  $\mathfrak{g}$  carries a Poisson bracket given by

$$\{F, G\}(\mu) = \left\langle \mu, \left[ \frac{\delta F}{\delta \mu}, \frac{\delta G}{\delta \mu} \right] \right\rangle$$

for  $\mu \in \mathfrak{g}^*$ , a formula found by Lie [1890], §75. As we saw in the Introduction, this *Lie–Poisson bracket* plays an important role in the Hamiltonian description of many physical systems. This bracket is not the bracket associated with any symplectic structure on  $\mathfrak{g}^*$ , but is an example of the more general concept of a *Poisson manifold*. However, the Lie–Poisson bracket *is* associated with a symplectic structure on coadjoint orbits and with the canonical symplectic structure on  $T^*G$ . These facts are developed in Chapters 13 and 14. Chapter 15 shows how this works in detail for the rigid body.

### 10.1 The Definition of Poisson Manifolds

This section generalizes the notion of a symplectic manifold by keeping just enough of the properties of Poisson brackets to describe Hamiltonian systems. The history of Poisson manifolds is complicated by the fact that the notion was rediscovered many times under different names; they occur in the works of Lie [1890], Dirac [1930], [1964], Pauli [1953], Martin [1959], Jost [1964], Arens [1970], Hermann [1973], Sudarshan and Mukunda [1974], Vinogradov and Krasilshchik [1975], and Lichnerowicz [1975b]. The name *Poisson manifold* was coined by Lichnerowicz. Further historical comments are given in §10.3

**Definition 10.1.1.** A **Poisson bracket** (or a **Poisson structure**) on a manifold  $P$  is a bilinear operation  $\{, \}$  on  $\mathcal{F}(P) = C^\infty(P)$  such that:

- (i)  $(\mathcal{F}(P), \{, \})$  is a Lie algebra; and
- (ii)  $\{, \}$  is a derivation in each factor, that is,

$$\{FG, H\} = \{F, H\}G + F\{G, H\},$$

for all  $F, G$ , and  $H \in \mathcal{F}(P)$ .

A manifold  $P$  endowed with a Poisson bracket on  $\mathcal{F}(P)$  is called a **Poisson manifold**.

A Poisson manifold is denoted by  $(P, \{, \})$  or simply by  $P$  if there is no danger of confusion. Note that any manifold has the **trivial Poisson structure** which is defined by setting  $\{F, G\} = 0$ , for all  $F, G \in \mathcal{F}(P)$ . Occasionally we consider two different Poisson brackets  $\{, \}_1$  and  $\{, \}_2$  on the same manifold; the two distinct Poisson manifolds are then denoted by  $(P, \{, \}_1)$  and  $(P, \{, \}_2)$ . The notation  $\{, \}_P$  for the bracket on  $P$  is also used when confusion might arise.

## Examples

**(a) Symplectic Bracket.** Any symplectic manifold is a Poisson manifold. The Poisson bracket is defined by the symplectic form as was shown in §5.5. Condition (ii) of the definition is satisfied as a consequence of the derivation property of vector fields:

$$\{FG, H\} = X_H[FG] = FX_H[G] + GX_H[F] = F\{G, H\} + G\{F, H\}. \quad \blacklozenge$$

**(b) Lie–Poisson Bracket.** If  $\mathfrak{g}$  is a Lie algebra, then its dual  $\mathfrak{g}^*$  is a Poisson manifold with respect to each of the **Lie–Poisson brackets**  $\{, \}_+$  and  $\{, \}_-$  defined by

$$\{F, G\}_\pm(\mu) = \pm \left\langle \mu, \left[ \frac{\delta F}{\delta \mu}, \frac{\delta G}{\delta \mu} \right] \right\rangle \quad (10.1.1)$$

for  $\mu \in \mathfrak{g}^*$  and  $F, G \in \mathcal{F}(\mathfrak{g}^*)$ . The properties of a Poisson bracket can be easily verified. Bilinearity and skew-symmetry are obvious. The derivation property of the bracket follows from the Leibniz rule for functional derivatives

$$\frac{\delta(FG)}{\delta \mu} = F(\mu) \frac{\delta G}{\delta \mu} + \frac{\delta F}{\delta \mu} G(\mu).$$

The Jacobi identity for the Lie–Poisson bracket follows from the Jacobi identity for the Lie algebra bracket and the formula

$$\begin{aligned} \pm \frac{\delta}{\delta \mu} \{F, G\}_{\pm} &= \left[ \frac{\delta F}{\delta \mu}, \frac{\delta G}{\delta \mu} \right] - \mathbf{D}^2 F(\mu) \left( \text{ad}_{\delta G / \delta \mu}^* \mu, \cdot \right) \\ &\quad + \mathbf{D}^2 G(\mu) \left( \text{ad}_{\delta F / \delta \mu}^* \mu, \cdot \right), \end{aligned} \quad (10.1.2)$$

where we recall from the preceding chapter that for each  $\xi \in \mathfrak{g}$ ,  $\text{ad}_{\xi} : \mathfrak{g} \rightarrow \mathfrak{g}$  denotes the map  $\text{ad}_{\xi}(\eta) = [\xi, \eta]$  and  $\text{ad}_{\xi}^* : \mathfrak{g}^* \rightarrow \mathfrak{g}^*$  is its dual. We give a different proof that (10.1.1) is a Poisson bracket in Chapter 13.  $\blacklozenge$

**(c) Rigid Body Bracket.** Specializing Example (b) to the Lie algebra of the rotation group,  $\mathfrak{so}(3) \cong \mathbb{R}^3$ , and identifying  $\mathbb{R}^3$  and  $(\mathbb{R}^3)^*$  via the standard inner product, we get the following Poisson structure on  $\mathbb{R}^3$ :

$$\{F, G\}_-(\mathbf{\Pi}) = -\mathbf{\Pi} \cdot (\nabla F \times \nabla G), \quad (10.1.3)$$

where  $\mathbf{\Pi} \in \mathbb{R}^3$  and  $\nabla F$ , the gradient of  $F$ , is evaluated at  $\mathbf{\Pi}$ . The Poisson bracket properties can be verified by direct computation in this case; see Exercise 1.2-1. We call (10.1.3) the *rigid body bracket*.  $\blacklozenge$

**(d) Ideal Fluid Bracket.** Specialize the Lie–Poisson bracket to the Lie algebra  $\mathfrak{X}_{\text{div}}(\Omega)$  of divergence-free vector fields defined in a region  $\Omega$  of  $\mathbb{R}^3$  and tangent to  $\partial\Omega$ , with the Lie bracket being the *negative* of the Jacobi–Lie bracket. Identify  $\mathfrak{X}_{\text{div}}^*(\Omega)$  with  $\mathfrak{X}_{\text{div}}(\Omega)$  using the  $L^2$  pairing

$$\langle \mathbf{v}, \mathbf{w} \rangle = \int_{\Omega} \mathbf{v} \cdot \mathbf{w} \, d^3x, \quad (10.1.4)$$

where  $\mathbf{v} \cdot \mathbf{w}$  is the ordinary dot product in  $\mathbb{R}^3$ . Thus, the *plus* Lie–Poisson bracket is

$$\{F, G\}(\mathbf{v}) = - \int_{\Omega} \mathbf{v} \cdot \left[ \frac{\delta F}{\delta \mathbf{v}}, \frac{\delta G}{\delta \mathbf{v}} \right] d^3x, \quad (10.1.5)$$

where the functional derivative  $\delta F / \delta \mathbf{v}$  is the element of  $\mathfrak{X}_{\text{div}}(\Omega)$  defined by

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} [F(\mathbf{v} + \varepsilon \delta \mathbf{v}) - F(\mathbf{v})] = \int_{\Omega} \frac{\delta F}{\delta \mathbf{v}} \cdot \delta \mathbf{v} \, d^3x. \quad \blacklozenge$$

**(e) Poisson–Vlasov Bracket.** Let  $(P, \{, \}_P)$  be a Poisson manifold and let  $\mathcal{F}(P)$  be the Lie algebra of functions under the Poisson bracket. Identify  $\mathcal{F}(P)^*$  with densities  $f$  on  $P$ . Then the Lie–Poisson bracket has the expression

$$\{F, G\}(f) = \int_P f \left\{ \frac{\delta F}{\delta f}, \frac{\delta G}{\delta f} \right\}_P. \quad (10.1.6)$$

$\blacklozenge$

**(f) Linearized Lie–Poisson Bracket.** Fix  $\nu \in \mathfrak{g}^*$  and define for any  $F, G \in \mathcal{F}(\mathfrak{g}^*)$  the bracket

$$\{F, G\}_{\pm}^{\nu}(\mu) = \pm \left\langle \nu, \left[ \frac{\delta F}{\delta \mu}, \frac{\delta G}{\delta \mu} \right] \right\rangle. \quad (10.1.7)$$

The properties of a Poisson bracket are verified as in the case of the Lie–Poisson bracket, the only difference being that (10.1.2) is replaced by

$$\pm \frac{\delta}{\delta \mu} \{F, G\}_{\pm}^{\nu} = -\mathbf{D}^2 F(\nu) \left( \text{ad}_{\delta G / \delta \mu}^* \mu, \cdot \right) + \mathbf{D}^2 G(\nu) \left( \text{ad}_{\delta F / \delta \mu}^* \mu, \cdot \right) \quad (10.1.8)$$

This bracket is useful in the description of the Lie–Poisson equations linearized at an equilibrium point, which is the reason for the name.<sup>1</sup> ♦

**(g) KdV Bracket.** Let  $S = [S^{ij}]$  be a symmetric matrix. On  $\mathcal{F}(\mathbb{R}^n, \mathbb{R}^n)$ , set

$$\{F, G\}(u) = \int_{-\infty}^{\infty} \sum_{i,j=1}^n S^{ij} \left[ \frac{\delta F}{\delta u^i} \frac{d}{dx} \left( \frac{\delta G}{\delta u^j} \right) - \frac{d}{dx} \left( \frac{\delta G}{\delta u^j} \right) \frac{\delta F}{\delta u^i} \right] dx \quad (10.1.9)$$

for functions  $F, G$  satisfying  $\delta F / \delta u$ , and  $\delta G / \delta u \rightarrow 0$  as  $x \rightarrow \pm\infty$ . This is a Poisson structure that is useful for the KdV equation and for gas dynamics (see Benjamin [1984]).<sup>2</sup> If  $S$  is invertible and  $S^{-1} = [S_{ij}]$ , then (10.1.9) is the Poisson bracket associated with the weak symplectic form

$$\begin{aligned} \Omega(u, v) = \frac{1}{2} \int_{-\infty}^{\infty} \sum_{i,j=1}^n S_{ij} \left[ \left( \int_{-\infty}^y u^i(x) dx \right) v^j(y) \right. \\ \left. - \left( \int_{-\infty}^y v^j(x) dx \right) u^i(y) \right] dy. \end{aligned} \quad (10.1.10)$$

This is easily seen by noting that  $X_H(u)$  is given by

$$X_H^i(u) = S^{ij} \frac{d}{dx} \frac{\delta H}{\delta u^j}. \quad \blacklozenge$$

**(h) Toda Lattice Bracket.** Let

$$P = \{(\mathbf{a}, \mathbf{b}) \in \mathbb{R}^{2n} \mid a^i > 0, i = 1, \dots, n\}$$

<sup>1</sup>See, for example, Abarbanel, Holm, Marsden, and Ratiu [1986].

<sup>2</sup>This is a particular case of Example (f), the Lie algebra being the pseudo-differential operators on the line of order  $\leq -1$  and  $\nu = Sd/dx$ .

and consider the bracket

$$\{F, G\}(\mathbf{a}, \mathbf{b}) = \left[ \left( \frac{\partial F}{\partial \mathbf{a}} \right)^T, \left( \frac{\partial F}{\partial \mathbf{b}} \right)^T \right] \mathbf{W} \begin{bmatrix} \frac{\partial G}{\partial \mathbf{a}} \\ \frac{\partial G}{\partial \mathbf{b}} \end{bmatrix}, \quad (10.1.11)$$

where  $(\partial F / \partial \mathbf{a})^T$  is the row vector  $(\partial F / \partial a^1, \dots, \partial F / \partial a^n)$ , etc., and

$$\mathbf{W} = \begin{bmatrix} 0 & \mathbf{A} \\ -\mathbf{A} & 0 \end{bmatrix}, \quad \text{where} \quad \mathbf{A} = \begin{bmatrix} a^1 & & 0 \\ & \ddots & \\ 0 & & a^n \end{bmatrix}. \quad (10.1.12)$$

In terms of the coordinate functions  $a_i, b_j$ , the bracket (10.1.11) is given by

$$\begin{aligned} \{a^i, a^j\} &= 0, \\ \{b^i, b^j\} &= 0, \\ \{a^i, b^j\} &= 0 \quad \text{if } i \neq j, \\ \{a^i, b^j\} &= a^i \quad \text{if } i = j. \end{aligned} \quad (10.1.13)$$

This Poisson bracket is determined by the symplectic form

$$\Omega = - \sum_{i=1}^n \frac{1}{a^i} da^i \wedge db^i \quad (10.1.14)$$

as an easy verification shows. The mapping  $(\mathbf{a}, \mathbf{b}) \mapsto (\log \mathbf{a}^{-1}, \mathbf{b})$  is a symplectic diffeomorphism of  $P$  with  $\mathbb{R}^{2n}$  endowed with the canonical symplectic structure. This symplectic structure is known as the *first Poisson structure of the non-periodic Toda lattice*. We shall not study this example in any detail in this book, but we point out that its bracket is the restriction of a Lie–Poisson bracket to a certain coadjoint orbit of the group of lower triangular matrices; we refer the interested reader to §14.5, Kostant [1979], and Symes [1980, 1982a,b] for further information. ♦

## Exercises

- ◇ **Exercise 10.1-1.** If  $P_1$  and  $P_2$  are Poisson manifolds, show how to make  $P_1 \times P_2$  into a Poisson manifold.
- ◇ **Exercise 10.1-2.** Verify directly that the Lie–Poisson bracket satisfies Jacobi’s identity.
- ◇ **Exercise 10.1-3 (A Quadratic Bracket).** Let  $A = [A^{ij}]$  be a skew-symmetric matrix. On  $\mathbb{R}^n$ , define  $B^{ij} = A^{ij}x^i x^j$  (no sum). Show that the following defines a Poisson structure:

$$\{F, G\} = \sum_{i,j=1}^n B^{ij} \frac{\partial F}{\partial x^i} \frac{\partial G}{\partial x^j}.$$

◇ **Exercise 10.1-4 (A Cubic Bracket).** For  $\mathbf{x} = (x^1, x^2, x^3) \in \mathbb{R}^3$ , put

$$\begin{aligned}\{x^1, x^2\} &= \|\mathbf{x}\|^2 x^3, \\ \{x^2, x^3\} &= \|\mathbf{x}\|^2 x^1, \\ \{x^3, x^1\} &= \|\mathbf{x}\|^2 x^2.\end{aligned}$$

Let  $B^{ij} = \{x^i, x^j\}$ , for  $i < j$  and  $i, j = 1, 2, 3$ , set  $B^{ji} = -B^{ij}$ , and define

$$\{F, G\} = \sum_{i,j=1}^n B^{ij} \frac{\partial F}{\partial x^i} \frac{\partial G}{\partial x^j}.$$

Check that this makes  $\mathbb{R}^3$  into a Poisson manifold.

◇ **Exercise 10.1-5.** Let  $\Phi : \mathfrak{g}^* \rightarrow \mathfrak{g}^*$  be a smooth function and define for  $F, H : \mathfrak{g}^* \rightarrow \mathbb{R}$ ,

$$\{F, H\}_{\Phi}(\mu) = \left\langle \Phi(\mu), \left[ \frac{\delta F}{\delta \mu}, \frac{\delta H}{\delta \mu} \right] \right\rangle.$$

(a) Show that this rule defines a Poisson bracket on  $\mathfrak{g}^*$  if and only if  $\Phi$  satisfies the following identity.

$$\begin{aligned}\langle \mathbf{D}\Phi(\mu) \cdot \text{ad}_{\zeta}^*(\mu), [\eta, \xi] \rangle &+ \langle \mathbf{D}\Phi(\mu) \cdot \text{ad}_{\eta}^* \Phi(\mu), [\xi, \zeta] \rangle \\ &+ \langle \mathbf{D}\Phi(\mu) \cdot \text{ad}_{\xi}^* \Phi(\mu), [\zeta, \eta] \rangle = 0,\end{aligned}$$

for all  $\xi, \eta, \zeta \in \mathfrak{g}$ , and all  $\mu \in \mathfrak{g}^*$ .

(b) Show that this relation holds if  $\Phi(\mu) = \mu$  and  $\Phi(\mu) = \nu$ , a fixed element of  $\mathfrak{g}^*$ , thereby obtaining the Lie–Poisson structure (10.1.1) and the linearized Lie–Poisson structure (10.1.7) on  $\mathfrak{g}^*$ . Show that it also holds if  $\Phi(\mu) = a\mu + \nu$  for some  $a \in \mathbb{R}$ .

(c) Assume  $\mathfrak{g}$  has a weakly nondegenerate invariant bilinear form  $\kappa : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{R}$  and identify  $\mathfrak{g}^*$  with  $\mathfrak{g}$  by  $\kappa$ . If  $\Psi : \mathfrak{g} \rightarrow \mathfrak{g}$  is smooth, show that

$$\{F, H\}_{\Psi}(\xi) = \kappa(\Psi(\xi), [\nabla F(\xi), \nabla H(\xi)])$$

is a Poisson bracket if and only if

$$\begin{aligned}\kappa(\mathbf{D}\Psi(\lambda) \cdot [\Psi(\lambda), \zeta], [\eta, \xi]) &+ \kappa(\mathbf{D}\Psi(\lambda) \cdot [\Psi(\lambda), \eta], [\xi, \zeta]) \\ &+ \kappa(\mathbf{D}\Psi(\lambda) \cdot [\Psi(\lambda), \xi], [\zeta, \eta]) = 0,\end{aligned}$$

for all  $\lambda, \xi, \eta, \zeta \in \mathfrak{g}$ . Here,  $\nabla F(\xi), \nabla H(\xi) \in \mathfrak{g}$  are the gradients of  $F$  and  $H$  at  $\xi \in \mathfrak{g}$  relative to  $\kappa$ .

Conclude as in (b) that this relation holds if  $\Psi(\lambda) = a\lambda + \chi$  for  $a \in \mathbb{R}$  and  $\chi \in \mathfrak{g}$ .

- (d) In the hypothesis of (c), let  $\Psi(\lambda) = \nabla\psi(\lambda)$  for some smooth  $\psi : \mathfrak{g} \rightarrow \mathbb{R}$ . Show that  $\{ , \}_\Psi$  is a Poisson bracket if and only if

$$\begin{aligned} & \mathbf{D}^2\psi(\lambda)([\nabla\psi(\lambda), \zeta], [\eta, \xi]) - \mathbf{D}^2\psi(\lambda)(\nabla\psi(\lambda), [\zeta, [\eta, \xi]]) \\ & + \mathbf{D}^2\psi(\lambda)([\nabla\psi(\lambda), \eta], [\xi, \zeta]) - \mathbf{D}^2\psi(\lambda)(\nabla\psi(\lambda), [\eta, [\xi, \zeta]]) \\ & + \mathbf{D}^2\psi(\lambda)([\nabla\psi(\lambda), \xi], [\zeta, \eta]) - \mathbf{D}^2\psi(\lambda)(\nabla\psi(\lambda), [\xi, [\zeta, \eta]]) = 0, \end{aligned}$$

for all  $\lambda, \xi, \eta, \zeta \in \mathfrak{g}$ . In particular, if  $\mathbf{D}^2\psi(\lambda)$  is an invariant bilinear form for all  $\lambda$ , this condition holds. However, if  $\mathfrak{g} = \mathfrak{so}(3)$  and  $\psi$  is arbitrary, then this condition also holds (see Exercise 1.3-2.)

## 10.2 Hamiltonian Vector Fields and Casimir Functions

**Hamiltonian Vector Fields.** We begin by extending the notion of a Hamiltonian vector field from the symplectic to the Poisson context.

**Proposition 10.2.1.** *Let  $P$  be a Poisson manifold. If  $H \in \mathcal{F}(P)$ , then there is a unique vector field  $X_H$  on  $P$  such that*

$$X_H[G] = \{G, H\}, \quad (10.2.1)$$

for all  $G \in \mathcal{F}(P)$ . We call  $X_H$  the **Hamiltonian vector field** of  $H$ .

**Proof.** This is a consequence of the fact that any derivation on  $\mathcal{F}(P)$  is represented by a vector field. Fixing  $H$ , the map  $G \mapsto \{G, H\}$  is a derivation, and so it uniquely determines  $X_H$  satisfying (10.3.1). (In infinite dimensions some technical conditions are needed for this proof, which are deliberately ignored here; see Abraham, Marsden, and Ratiu [1988], §4.2.) ■

Notice that (10.2.1) agrees with our definition of Poisson brackets in the symplectic case, so if the Poisson manifold  $P$  is symplectic,  $X_H$  defined here agrees with the definition in §5.5.

**Proposition 10.2.2.** *The map  $H \mapsto X_H$  of  $\mathcal{F}(P)$  to  $\mathfrak{X}(P)$  is a Lie algebra antihomomorphism; that is,*

$$[X_H, X_K] = -X_{\{H, K\}}.$$

**Proof.** Using Jacobi's identity, we find that

$$\begin{aligned} [X_H, X_K][F] &= X_H[X_K[F]] - X_K[X_H[F]] \\ &= \{\{F, K\}, H\} - \{\{F, H\}, K\} \\ &= -\{F, \{H, K\}\} \\ &= -X_{\{H, K\}}[F]. \end{aligned} \quad \blacksquare$$

**Poisson Bracket Form.** Next, we establish the equation  $\dot{F} = \{F, H\}$  in the Poisson context.

**Proposition 10.2.3.** *Let  $\varphi$  be a flow on a Poisson manifold  $P$ . Then*

(i) *for any  $F \in \mathcal{F}(U)$ ,  $U$  open in  $P$ ,*

$$\frac{d}{dt}(F \circ \varphi_t) = \{F, H\} \circ \varphi_t = \{F \circ \varphi_t, H\},$$

*or, for short,*

$$\dot{F} = \{F, H\}, \quad \text{for any } F \in \mathcal{F}(U), U \text{ open in } P,$$

*if and only if  $\varphi_t$  is the flow of  $X_H$ .*

(ii) *If  $\varphi_t$  is the flow of  $X_H$ , then  $H \circ \varphi_t = H$ .*

**Proof.** (i) Let  $z \in P$ . Then

$$\frac{d}{dt}F(\varphi_t(z)) = \mathbf{d}F(\varphi_t(z)) \cdot \frac{d}{dt}\varphi_t(z)$$

and

$$\{F, H\}(\varphi_t(z)) = \mathbf{d}F(\varphi_t(z)) \cdot X_H(\varphi_t(z)).$$

The two expressions are equal for any  $F \in \mathcal{F}(U)$ ,  $U$  open in  $P$ , if and only if

$$\frac{d}{dt}\varphi_t(z) = X_H(\varphi_t(z)),$$

by the Hahn–Banach theorem. This is equivalent to  $t \mapsto \varphi_t(z)$  being the integral curve of  $X_H$  with initial condition  $z$ , that is,  $\varphi_t$  is the flow of  $X_H$ .

On the other hand, if  $\varphi_t$  is the flow of  $X_H$ , then we have

$$X_H(\varphi_t(z)) = T_z\varphi_t(X_H(z))$$

so that by the chain rule

$$\begin{aligned} \frac{d}{dt}F(\varphi_t(z)) &= \mathbf{d}F(\varphi_t(z)) \cdot X_H(\varphi_t(z)) \\ &= \mathbf{d}F(\varphi_t(z)) \cdot T_z\varphi_t(X_H(z)) \\ &= \mathbf{d}(F \circ \varphi_t)(z) \cdot X_H(z) \\ &= \{F \circ \varphi_t, H\}(z). \end{aligned}$$

(ii) For the proof of (ii), let  $H = F$  in (i). ■

**Corollary 10.2.4.** *Let  $G, H \in \mathcal{F}(P)$ . Then  $G$  is constant along the integral curves of  $X_H$  if and only if  $\{G, H\} = 0$ , if and only if  $H$  is constant along the integral curves of  $X_G$ .*



Among the elements of  $\mathcal{F}(P)$  are functions  $C$  such that  $\{C, F\} = 0$ , for all  $F \in \mathcal{F}(P)$ , that is,  $C$  is constant along the flow of all Hamiltonian vector fields or, equivalently,  $X_C = 0$ , that is,  $C$  generates trivial dynamics. Such functions are called **Casimir functions** of the Poisson structure. They form the center of the Poisson algebra.<sup>3</sup> This terminology is used in, for example, Sudarshan and Mukunda [1974]. H. B. G. Casimir is a prominent physicist who wrote his thesis (Casimir [1931]) on the quantum mechanics of the rigid body, under the direction of Paul Ehrenfest. Recall that it was Ehrenfest who, in *his* thesis, worked on the variational structure of ideal flow in Lagrangian or material representation.

**Some History of Poisson Structures.**<sup>4</sup> Following from the work of Lagrange and Poisson discussed at the end of §8.1, the general concept of a Poisson manifold should be credited to Sophus Lie in his treatise on transformation groups written around 1880 in the chapter on “function groups.” Lie uses the word “group” and “algebra.” For example, a “function group” should really be translated as “function algebra.” On page 237, Lie defines what today is called a Poisson structure. The title of Chapter 19 is *The Coadjoint Group*, which is explicitly identified on page 334. Chapter 17, pages 294–298, defines a linear Poisson structure on the dual of a Lie algebra, today called the Lie–Poisson structure, and “Lie’s Third Theorem” is proved for the set of regular elements. On page 349, together with a remark on page 367, it is shown that the Lie–Poisson structure naturally induces a symplectic structure on each coadjoint orbit. As we shall point out in §11.2, Lie also had many of the ideas of momentum maps. For many years this work appears to have been forgotten.

Because of the above history, Marsden and Weinstein [1983] coined the phrase “Lie–Poisson bracket” for this object, and this terminology is now in common use. However, it is not clear that Lie understood the fact that the Lie–Poisson bracket is obtained by a simple reduction process, namely, that it is induced from the canonical cotangent Poisson bracket on  $T^*G$  by passing to  $\mathfrak{g}^*$  regarded as the quotient  $T^*G/G$ , as will be explained in Chapter 13. The link between the closedness of the symplectic form and the Jacobi identity is a little harder to trace explicitly; some comments in this direction are given in Souriau [1970], who gives credit to Maxwell.

Lie’s work starts by taking functions  $F_1, \dots, F_r$  on a symplectic manifold  $M$ , with the property that there exist functions  $G_{ij}$  of  $r$  variables, such that

$$\{F_i, F_j\} = G_{ij}(F_1, \dots, F_r).$$

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<sup>3</sup>The *center* of a group or algebra is the set of elements that commute with all elements of the algebra.

<sup>4</sup>We thank Hans Duistermaat and Alan Weinstein for their help with the comments in this section; the paper of Weinstein [1983a] should also be consulted by the interested reader.

In Lie's time, all functions in sight are implicitly assumed to be analytic. The collection of all functions  $\phi$  of  $F_1, \dots, F_r$  is the "function group"; it is provided with the bracket

$$[\phi, \psi] = \sum_{ij} G_{ij} \phi_i \psi_j, \quad (10.2.2)$$

where

$$\phi_i = \frac{\partial \phi}{\partial F_i} \quad \text{and} \quad \psi_j = \frac{\partial \psi}{\partial F_j}.$$

Considering  $F = (F_1, \dots, F_r)$  as a map from  $M$  to an  $r$ -dimensional space  $P$ , and  $\phi$  and  $\psi$  as functions on  $P$ , one may formulate this as:  $[\phi, \psi]$  is a Poisson structure on  $P$ , with the property that

$$F^*[\phi, \psi] = \{F^*\phi, F^*\psi\}.$$

Lie writes down the equations for the  $G_{ij}$  that follow from the antisymmetry and the Jacobi identity for the bracket  $\{, \}$  on  $M$ . He continues with the question: if a given a system of functions  $G_{ij}$  in  $r$  variables satisfy these equations, is it induced, as above, from a function group of functions of  $2n$  variables? He shows that under suitable rank conditions the answer is yes. As we shall see below, this result is the precursor to many of the fundamental results about the geometry of Poisson manifolds.

It is obvious that if  $G_{ij}$  is a system that satisfies the equations that Lie writes down, then (10.2.2) is a Poisson structure in  $r$ -dimensional space. Vice versa, for any Poisson structure  $[\phi, \psi]$ , the functions

$$G_{ij} = [F_i, F_j]$$

satisfy Lie's equations.

Lie continues with more remarks on local normal forms of function groups, (i.e., of Poisson structures), under suitable rank conditions, which are not always stated as explicitly as one would like. These amount to: a Poisson structure of constant rank is the same as a foliation with symplectic leaves. It is this characterization that Lie uses to get the symplectic form on the coadjoint orbits. On the other hand, Lie does not apply the symplectic form on the coadjoint orbits to representation theory. Representation theory of Lie groups started only later with Schur on  $GL(n)$ , and was continued by Elie Cartan with representations of semisimple Lie algebras, and in the 1930s, by Weyl with the representation of compact Lie groups. The coadjoint orbit symplectic structure was connected with representation theory in the work of Kirillov and Kostant. On the other hand, Lie *did* apply the Poisson structure on the dual of the Lie algebra to prove that every abstract Lie algebra can be realized as a Lie algebra of Hamiltonian vector fields, or as a Lie subalgebra of the Poisson algebra of functions on some symplectic

manifold. This is “Lie’s third fundamental theorem” in the form given by Lie.

Of course, in geometry, people like Engel, Study and, in particular, Elie Cartan, studied Lie’s work intensely and propagated it very actively. However, through the tainted glasses of retrospection, Lie’s work on Poisson structures did not appear to receive as much attention in mechanics as it deserved; for example, even though Cartan himself did very important work in mechanics (such as, Cartan [1923, 1928a,b]), he did not seem to realize that the Lie–Poisson bracket was central to the Hamiltonian description of some of the rotating fluid systems he was studying. However, others, such as Hamel [1904, 1949], did study Lie intensively and used it to make substantial contributions and extensions (such as to the study of nonholonomic systems, including rolling constraints), but many other active schools seem to have missed it. Even more surprising in this context is the contribution of Poincaré [1901b, 1910] to the Lagrangian side of the story, a tale that we shall come to in Chapter 13.

## Examples

**(a) Symplectic Case.** On a symplectic manifold  $P$ , any Casimir function is constant on connected components of  $P$ . This holds since in the symplectic case,  $X_C = 0$  implies  $\mathbf{d}C = 0$  and hence  $C$  is locally constant. ♦

**(b) Rigid Body Casimirs.** In the context of Example (c) of §10.2, let  $C(\mathbf{\Pi}) = \|\mathbf{\Pi}\|^2/2$ . Then  $\nabla C(\mathbf{\Pi}) = \mathbf{\Pi}$  and by the properties of the triple product, we have for any  $F \in \mathcal{F}(\mathbb{R}^3)$ ,

$$\begin{aligned}\{C, F\}(\mathbf{\Pi}) &= -\mathbf{\Pi} \cdot (\nabla C \times \nabla F) = -\mathbf{\Pi} \cdot (\mathbf{\Pi} \times \nabla F) \\ &= -\nabla F \cdot (\mathbf{\Pi} \times \mathbf{\Pi}) = 0.\end{aligned}$$

This shows that  $C(\mathbf{\Pi}) = \|\mathbf{\Pi}\|^2/2$  is a Casimir function. A similar argument shows that

$$C_\Phi(\mathbf{\Pi}) = \Phi\left(\frac{1}{2}\|\mathbf{\Pi}\|^2\right) \quad (10.2.3)$$

is a Casimir function, where  $\Phi$  is an arbitrary (differentiable) function of one variable; this is proved by noting that

$$\nabla C_\Phi(\mathbf{\Pi}) = \Phi'\left(\frac{1}{2}\|\mathbf{\Pi}\|^2\right) \mathbf{\Pi}. \quad \blacklozenge$$

**(c) Helicity.** In Example (d) of §10.2, the *helicity*

$$C(\mathbf{v}) = \int_{\Omega} \mathbf{v} \cdot (\nabla \times \mathbf{v}) d^3x \quad (10.2.4)$$

can be checked to be a Casimir function if  $\partial\Omega = \emptyset$ . ♦

..... 11 March 1998—16h24 .....

**(d) Poisson–Vlasov Casimirs.** In Example (e) of §10.2, given a differentiable function  $\Phi : \mathbb{R} \rightarrow \mathbb{R}$ , the map  $C : \mathcal{F}(P) \rightarrow \mathbb{R}$  defined by

$$C(f) = \int \Phi(f(q, p)) dq dp \quad (10.2.5)$$

is a Casimir function. Here we choose  $P$  to be symplectic, have written  $dq dp = dz$  for the Liouville measure, and have used it to identify functions and densities.  $\blacklozenge$

### Exercises

- ◇ **Exercise 10.2-1.** Verify the relation  $[X_H, X_K] = -X_{\{H, K\}}$  directly for the rigid body bracket.
- ◇ **Exercise 10.2-2.** Verify that (10.2.5):

$$C(f) = \int \Phi(f(q, p)) dq dp,$$

defines a Casimir function.

- ◇ **Exercise 10.2-3.** Let  $P$  be a Poisson manifold and let  $M \subset P$  be a connected submanifold with the property that for each  $v \in T_x M$  there is a Hamiltonian vector field  $X_H$  on  $P$  such that  $v = X_H(x)$ ; that is,  $T_x M$  is spanned by Hamiltonian vector fields. Prove that any Casimir function is constant on  $M$ .

## 10.3 Properties of Hamiltonian Flows

**Hamiltonian Flows Are Poisson.** Now we establish the Poisson analogue of the symplectic nature of the flows of Hamiltonian vector fields.

**Proposition 10.3.1.** *If  $\varphi_t$  is the flow of  $X_H$ , then*

$$\varphi_t^* \{F, G\} = \{\varphi_t^* F, \varphi_t^* G\};$$

*in other words,*

$$\{F, G\} \circ \varphi_t = \{F \circ \varphi_t, G \circ \varphi_t\}.$$

*Thus, the flows of Hamiltonian vector fields preserve the Poisson structure.*

**Proof.** This is actually true even for time-dependent Hamiltonian systems (as we will see later), but here we will prove it only in the time-independent case. Let  $F, K \in \mathcal{F}(P)$  and let  $\varphi_t$  be the flow of  $X_H$ . Let

$$u = \{F \circ \varphi_t, K \circ \varphi_t\} - \{F, K\} \circ \varphi_t.$$

..... 11 March 1998—16h24 .....

Because of the bilinearity of the Poisson bracket,

$$\frac{du}{dt} = \left\{ \frac{d}{dt} F \circ \varphi_t, K \circ \varphi_t \right\} + \left\{ F \circ \varphi_t, \frac{d}{dt} K \circ \varphi_t \right\} - \frac{d}{dt} \{F, K\} \circ \varphi_t.$$

Using Proposition 10.2.3, this becomes

$$\frac{du}{dt} = \{ \{F \circ \varphi_t, H\}, K \circ \varphi_t \} + \{ F \circ \varphi_t, \{K \circ \varphi_t, H\} \} - \{ \{F, K\} \circ \varphi_t, H \},$$

which, by Jacobi's identity, gives

$$\frac{du}{dt} = \{u, H\} = X_H[u].$$

The unique solution of this equation is  $u_t = u_0 \circ \varphi_t$ . Since  $u_0 = 0$ , we get  $u = 0$ , which is the result. ■

As in the symplectic case, with which this is of course consistent, this argument shows how Jacobi's identity plays a crucial role.

**Poisson Maps.** A smooth mapping  $f : P_1 \rightarrow P_2$  between the two Poisson manifolds  $(P_1, \{, \}_1)$  and  $(P_2, \{, \}_2)$  is called **canonical** or **Poisson** if

$$f^* \{F, G\}_2 = \{f^* F, f^* G\}_1,$$

for all  $F, G \in \mathcal{F}(P_2)$ . Proposition 10.3.1 shows that flows of Hamiltonian vector fields are canonical maps. We saw already in Chapter 5 that if  $P_1$  and  $P_2$  are symplectic manifolds, a map  $f : P_1 \rightarrow P_2$  is canonical if and only if it is symplectic.

**Coordinate representation.** We have seen that the matrix  $[B^{IJ}]$  of the Poisson tensor  $B$  converts the differential

$$\mathbf{d}H = \frac{\partial H}{\partial z^I} dz^I$$

of a function to the corresponding Hamiltonian vector field; this is consistent with our treatment in the Introduction and Overview. Another basic concept, that of a Poisson map, is also worthwhile working out in coordinates.

Let  $f : P_1 \rightarrow P_2$  be a Poisson map, so  $\{F \circ f, G \circ f\}_1 = \{F, G\}_2 \circ f$ . In coordinates  $z^I$  on  $P_1$  and  $w^K$  on  $P_2$ , and writing  $w^K = w^K(z^I)$  for the map  $f$ , this reads

$$\frac{\partial}{\partial z^I} (F \circ f) \frac{\partial}{\partial z^J} (G \circ f) B_1^{IJ}(z) = \frac{\partial F}{\partial w^K} \frac{\partial G}{\partial w^L} B_2^{KL}(w).$$

By the chain rule, this is equivalent to

$$\frac{\partial F}{\partial w^K} \frac{\partial w^K}{\partial z^I} \frac{\partial G}{\partial w^L} \frac{\partial w^L}{\partial z^J} B_1^{IJ}(z) = \frac{\partial F}{\partial w^K} \frac{\partial G}{\partial w^L} B_2^{KL}(w).$$

..... 11 March 1998—16h24 .....

Since  $F$  and  $G$  are arbitrary,  $f$  is Poisson iff

$$B_1^{IJ}(z) \frac{\partial w^K}{\partial z^I} \frac{\partial w^L}{\partial z^J} = B_2^{KL}(w).$$

Intrinsically, regarding  $B_1(z)$  as a map  $B_1(z) : T_z^*P_1 \times T_z^*P_1 \rightarrow \mathbb{R}$ , this reads

$$B_1(z)(T_z^*f \cdot \alpha_w, T_z^*f \cdot \beta_w) = B_2(w)(\alpha_w, \beta_w), \quad (10.3.1)$$

where  $\alpha_w, \beta_w \in T_w^*P_2$  and  $f(z) = w$ . In analogy with the case of vector fields we shall say that if (10.4.5) holds, then  $B_1$  and  $B_2$  are ***f-related*** and denote it by  $B_1 \sim_f B_2$ . In other words,  $f$  is Poisson iff

$$B_1 \sim_f B_2. \quad (10.3.2)$$

**Properties of Poisson Maps.** The next proposition shows that Poisson maps push Hamiltonian flows to Hamiltonian flows.

**Proposition 10.3.2.** *Let  $f : P_1 \rightarrow P_2$  be a Poisson map and let  $H \in \mathcal{F}(P_2)$ . If  $\varphi_t$  is the flow of  $X_H$  and  $\psi_t$  is the flow of  $X_{H \circ f}$ , then*

$$\varphi_t \circ f = f \circ \psi_t \quad \text{and} \quad Tf \circ X_{H \circ f} = X_H \circ f.$$

*Conversely, if  $f$  is a map from  $P_1$  to  $P_2$  and for any  $H \in \mathcal{F}(P_2)$ , the Hamiltonian vector fields  $X_{H \circ f} \in \mathfrak{X}(P_1)$  and  $X_H \in \mathfrak{X}(P_2)$  are  $f$ -related, that is,*

$$Tf \circ X_{H \circ f} = X_H \circ f,$$

*then  $f$  is canonical.*

**Proof.** For any  $G \in \mathcal{F}(P_2)$  and  $z \in P_1$ , Proposition 10.2.3(i) and the definition of Poisson maps yield

$$\begin{aligned} \frac{d}{dt} G((f \circ \psi_t)(z)) &= \frac{d}{dt} (G \circ f)(\psi_t(z)) \\ &= \{G \circ f, H \circ f\}(\psi_t(z)) = \{G, H\}(f \circ \psi_t)(z), \end{aligned}$$

that is,  $(f \circ \psi_t)(z)$  is an integral curve of  $X_H$  on  $P_2$  through the point  $f(z)$ . Since  $(\varphi_t \circ f)(z)$  is another such curve, uniqueness of integral curves implies that

$$(f \circ \psi_t)(z) = (\varphi_t \circ f)(z).$$

The relation  $Tf \circ X_{H \circ f} = X_H \circ f$  follows from  $f \circ \psi_t = \varphi_t \circ f$  by taking the time-derivative.

Conversely, assume that for any  $H \in \mathcal{F}(P_2)$  we have  $Tf \circ X_{H \circ f} = X_H \circ f$ . Therefore, by the chain rule,

$$\begin{aligned} X_{H \circ f} [F \circ f](z) &= \mathbf{d}F(f(z)) \cdot T_z f(X_{H \circ f}(z)) \\ &= \mathbf{d}F(f(z)) \cdot X_H(f(z)) = X_H [F](f(z)), \end{aligned}$$

that is,  $X_{H \circ f} [f^* F] = f^* (X_H [F])$ . Thus, for  $G \in \mathcal{F}(P_2)$ ,

$$\{G, H\} \circ f = f^* (X_H [G]) = X_{H \circ f} [f^* G] = \{G \circ f, H \circ f\}$$

and so  $f$  is canonical. ■

### Exercises

- ◇ **Exercise 10.3-1.** Verify directly that a rotation  $R : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  is a Poisson map for the rigid body bracket.
- ◇ **Exercise 10.3-2.** If  $P_1$  and  $P_2$  are Poisson manifolds, show that the projection  $\pi_1 : P_1 \times P_2 \rightarrow P_1$  is a Poisson map. Is the corresponding statement true for symplectic maps?

## 10.4 The Poisson Tensor

**Definition of the Poisson Tensor.** By the derivation property of the Poisson bracket, the value of the bracket  $\{F, G\}$  at  $z \in P$  (and thus  $X_F(z)$  as well), depends on  $F$  only through  $\mathbf{d}F(z)$  (see Abraham, Marsden, and Ratiu [1988], Theorem 4.2.16 for this type of argument). Thus, there is a contravariant antisymmetric two-tensor

$$B : T^*P \times T^*P \rightarrow \mathbb{R}$$

such that

$$B(z)(\alpha_z, \beta_z) = \{F, G\}(z),$$

where  $\mathbf{d}F(z) = \alpha_z$  and  $\mathbf{d}G(z) = \beta_z \in T_z^*P$ . This tensor  $B$  is called a **cosymplectic** or **Poisson structure**. In local coordinates  $(z^1, \dots, z^n)$ ,  $B$  is determined by its matrix elements  $\{z^I, z^J\} = B^{IJ}(z)$  and the bracket becomes

$$\{F, G\} = B^{IJ}(z) \frac{\partial F}{\partial z^I} \frac{\partial G}{\partial z^J}. \quad (10.4.1)$$

Let  $B^\sharp : T^*P \rightarrow TP$  be the vector bundle map associated to  $B$ , that is,

$$B(z)(\alpha_z, \beta_z) = \langle \alpha_z, B^\sharp(z)(\beta_z) \rangle.$$

Consistent with our conventions  $\dot{F} = \{F, H\}$ , the Hamiltonian vector field is given by  $X_H(z) = B_z^\sharp \cdot \mathbf{d}H(z)$ . Indeed,  $\dot{F}(z) = \mathbf{d}F(z) \cdot X_H(z)$  and

$$\{F, H\}(z) = B(z)(\mathbf{d}F(z), \mathbf{d}H(z)) = \langle \mathbf{d}F(z), B^\sharp(z)(\mathbf{d}H(z)) \rangle.$$

Comparing these expressions gives the stated result.

..... 11 March 1998—16h24 .....

**Coordinate Representation.** A convenient way to specify a bracket in finite dimensions is by giving the coordinate relations  $\{z^I, z^J\} = B^{IJ}(z)$ . The Jacobi identity is then implied by the special cases

$$\{\{z^I, z^J\}, z^K\} + \{\{z^K, z^I\}, z^J\} + \{\{z^J, z^K\}, z^I\} = 0,$$

which are equivalent to the differential equations

$$B^{LI} \frac{\partial B^{JK}}{\partial z^L} + B^{LJ} \frac{\partial B^{KI}}{\partial z^L} + B^{LK} \frac{\partial B^{IJ}}{\partial z^L} = 0 \quad (10.4.2)$$

(the terms are cyclic in  $I, J, K$ ). Writing  $X_H[F] = \{F, H\}$  in coordinates gives

$$X_H^I \frac{\partial F}{\partial z^I} = B^{JK} \frac{\partial F}{\partial z^J} \frac{\partial H}{\partial z^K}$$

and so

$$X_H^I = B^{IJ} \frac{\partial H}{\partial z^J}. \quad (10.4.3)$$

This expression tells us that  $B^{IJ}$  should be thought of as the negative inverse of the symplectic matrix, which is literally correct in the nondegenerate case. Indeed, if we write out

$$\Omega(X_H, v) = \mathbf{d}H \cdot v$$

in coordinates, we get

$$\Omega_{IJ} X_H^I v^J = \frac{\partial H}{\partial z^J} v^J, \quad \text{i.e.,} \quad \Omega_{IJ} X_H^I = \frac{\partial H}{\partial z^J}.$$

If  $[\Omega^{IJ}]$  denotes the inverse of  $[\Omega_{IJ}]$ , we get

$$X_H^I = \Omega^{JI} \frac{\partial H}{\partial z^J}, \quad (10.4.4)$$

so comparing (10.4.3) and (10.4.4) we see that

$$B^{IJ} = -\Omega^{IJ}.$$

Recalling that the matrix of  $\Omega^\sharp$  is the inverse of that of  $\Omega^\flat$  and that the matrix of  $\Omega^\flat$  is the *negative* of that of  $\Omega$ , we see that  $B^\sharp = \Omega^\sharp$ .

Let us prove this abstractly. The basic link between the Poisson tensor  $B$  and the symplectic form  $\Omega$  is that they give the same Poisson bracket:

$$\{F, H\} = B(\mathbf{d}F, \mathbf{d}H) = \Omega(X_F, X_H),$$

that is,

$$\langle \mathbf{d}F, B^\sharp \mathbf{d}H \rangle = \langle \mathbf{d}F, X_H \rangle.$$

..... 11 March 1998—16h24 .....



But

$$\Omega(X_H, v) = \mathbf{d}H \cdot v,$$

and so

$$\langle \Omega^\flat X_H, v \rangle = \langle \mathbf{d}H, v \rangle,$$

whence,

$$X_H = \Omega^\sharp \mathbf{d}H$$

since  $\Omega^\sharp = (\Omega^\flat)^{-1}$ . Thus,  $B^\sharp \mathbf{d}H = \Omega^\sharp \mathbf{d}H$ , for all  $H$ , and thus,

$$B^\sharp = \Omega^\sharp.$$

**Lie Derivative of the Poisson Tensor.** The next Proposition is closely related to the fact that the flows of Hamiltonian vector fields are Poisson maps. We return to these points in §10.6.

**Proposition 10.4.1.** *For any function  $H \in \mathcal{F}(P)$ , we have  $\mathcal{L}_{X_H} B = 0$ .*

**Proof.** By definition, we have

$$B(\mathbf{d}F, \mathbf{d}G) = \{F, G\} = X_G[F]$$

for any locally defined functions  $F$  and  $G$  on  $P$ . Therefore,

$$\mathcal{L}_{X_H}(B(\mathbf{d}F, \mathbf{d}G)) = \mathcal{L}_{X_H} \{F, G\} = \{\{F, G\}, H\}.$$

However, since the Lie derivative is a derivation,

$$\begin{aligned} \mathcal{L}_{X_H}(B(\mathbf{d}F, \mathbf{d}G)) &= (\mathcal{L}_{X_H} B)(\mathbf{d}F, \mathbf{d}G) + B(\mathcal{L}_{X_H} \mathbf{d}F, \mathbf{d}G) + B(\mathbf{d}F, \mathcal{L}_{X_H} \mathbf{d}G) \\ &= (\mathcal{L}_{X_H} B)(\mathbf{d}F, \mathbf{d}G) + B(\mathbf{d}\{F, H\}, \mathbf{d}G) + B(\mathbf{d}F, \mathbf{d}\{G, H\}) \\ &= (\mathcal{L}_{X_H} B)(\mathbf{d}F, \mathbf{d}G) + \{\{F, H\}, G\} + \{F, \{G, H\}\} \\ &= (\mathcal{L}_{X_H} B)(\mathbf{d}F, \mathbf{d}G) + \{\{F, G\}, H\}, \end{aligned}$$

by the Jacobi identity. It follows that  $(\mathcal{L}_{X_H} B)(\mathbf{d}F, \mathbf{d}G) = 0$  for any locally defined functions  $F, G \in \mathcal{F}(U)$ . Since any element of  $T_z^*P$  can be written as  $\mathbf{d}F(z)$  for some  $F \in \mathcal{F}(U)$ ,  $U$  open in  $P$ , it follows that  $\mathcal{L}_{X_H} B = 0$ . ■

**Pauli–Jost Theorem.** Suppose that the Poisson tensor  $B$  is strongly nondegenerate, that is, it defines an isomorphism  $B^\sharp : \mathbf{d}F(z) \mapsto X_F(z)$  of  $T_z^*P$  with  $T_zP$ , for all  $z \in P$ . Then  $P$  is symplectic and the symplectic form  $\Omega$  is defined by the formula  $\Omega(X_F, X_G) = \{F, G\}$  for any locally defined Hamiltonian vector fields  $X_F$  and  $X_G$ . One gets  $\mathbf{d}\Omega = 0$  from Jacobi's

identity—see Exercise 5.5-1. This is the **Pauli–Jost Theorem**, due to Pauli [1953] and Jost [1964].

One may be tempted to formulate the above nondegeneracy assumption in a slightly weaker form involving only the Poisson bracket: *suppose that for every open subset  $V$  of  $P$ , if  $F \in \mathcal{F}(V)$  and  $\{F, G\} = 0$  for all  $G \in \mathcal{F}(U)$  and all open subsets  $U$  of  $V$ , then  $\mathbf{d}F = 0$  on  $V$ , that is,  $F$  is constant on the connected components of  $V$ .* This condition does *not* imply that  $P$  is a symplectic as the following counter example shows. Let  $P = \mathbb{R}^2$  with Poisson bracket.

$$\{F, G\}(x, y) = y \left( \frac{\partial F}{\partial x} \frac{\partial G}{\partial y} - \frac{\partial F}{\partial y} \frac{\partial G}{\partial x} \right).$$

If  $\{F, G\} = 0$ , for all  $G$ , then  $F$  must be constant on both the upper and lower half-planes and hence by continuity it must be constant on  $\mathbb{R}^2$ . However,  $\mathbb{R}^2$  with this Poisson structure is clearly not symplectic.

**Characteristic Distribution.** The subset  $B^\sharp(T^*P)$  of  $TP$  is called the **characteristic field** of the Poisson structure; it need not be a subbundle of  $TP$ , in general. Note that skew-symmetry of the tensor  $B$  is equivalent to  $(B^\sharp)^* = -B^\sharp$ , where  $(B^\sharp)^* : T^*P \rightarrow TP$  is the dual of  $B^\sharp$ . If  $P$  is finite dimensional, the **rank** of the Poisson structure at a point  $z \in P$  is defined to be the rank of  $B^\sharp(z) : T_z^*P \rightarrow T_zP$ ; in local coordinates, it is the rank of the matrix  $[B^{IJ}(z)]$ . Since the flows of Hamiltonian vector fields preserve the Poisson structure, the rank is constant along such a flow. A Poisson structure for which the rank is everywhere equal to the dimension of the manifold is nondegenerate and hence symplectic.

**Poisson Immersions and Submanifolds.** An injectively immersed submanifold  $i : S \rightarrow P$  is called a **Poisson immersion** if any Hamiltonian vector field defined on an open subset of  $P$  containing  $i(S)$  is in the range of  $T_z i$  at all points  $i(z)$  for  $z \in S$ . This is equivalent to the following assertion:

**Proposition 10.4.2.** *An immersion  $i : S \rightarrow P$  is Poisson iff it satisfies the following condition. If  $F, G : V \subset S \rightarrow \mathbb{R}$ , where  $V$  is open in  $S$ , and if  $\overline{F}, \overline{G} : U \rightarrow \mathbb{R}$  are extensions of  $F \circ i^{-1}, G \circ i^{-1} : i(V) \rightarrow \mathbb{R}$  to an open neighborhood  $U$  of  $i(V)$  in  $P$ , then  $\{\overline{F}, \overline{G}\}|_{i(V)}$  is well defined and independent of the extensions. The immersed submanifold  $S$  is thus endowed with an induced Poisson structure and  $i : S \rightarrow P$  becomes a Poisson map.*

**Proof.** If  $i : S \rightarrow P$  is an injectively immersed Poisson manifold, then

$$\begin{aligned} \{\overline{F}, \overline{G}\}(i(z)) &= \mathbf{d}\overline{F}(i(z)) \cdot X_{\overline{G}}(i(z)) = \mathbf{d}\overline{F}(i(z)) \cdot T_z i(v) \\ &= \mathbf{d}(\overline{F} \circ i)(z) \cdot v = \mathbf{d}F(z) \cdot v, \end{aligned}$$

..... 11 March 1998—16h24 .....

where  $v \in T_z S$  is the unique vector satisfying  $X_{\overline{G}}(i(z)) = T_z i(v)$ . Thus,  $\{\overline{F}, \overline{G}\}(i(z))$  is independent on the extension  $\overline{F}$  of  $F \circ i^{-1}$ . By skew-symmetry of the bracket, it is also independent on the extension  $\overline{G}$  of  $G \circ i^{-1}$ . Then one can define a Poisson structure on  $S$  by setting

$$\{F, G\} = \{\overline{F}, \overline{G}\}|i(V)$$

for any open subset  $V$  of  $S$ . In this way  $i : S \rightarrow P$  becomes a Poisson map since by the computation above we have  $X_{\overline{G}}(i(z)) = T_z i(X_G)$

Conversely, assume that the condition on the bracket stated above holds and let  $H : U \rightarrow P$  be a Hamiltonian defined on an open subset  $U$  of  $P$  intersecting  $i(S)$ . Then, by what was already shown,  $S$  is a Poisson manifold and  $i : S \rightarrow P$  is a Poisson map. We claim that if  $z \in S$  is such that  $i(z) \in U$ , we have

$$X_H(i(z)) = T_z i(X_{H \circ i}(z)),$$

and thus  $X_H(i(z)) \in \text{range } T_z i$ , thereby showing that  $i : S \rightarrow P$  is a Poisson immersion. To see this, let  $K : U \rightarrow \mathbb{R}$  be an arbitrary function. We have

$$\begin{aligned} \mathbf{d}K(i(z)) \cdot X_H(i(z)) &= \{K, H\}(i(z)) = \{K \circ i, H \circ i\}(z) \\ &= \mathbf{d}(K \circ i)(z) \cdot X_{H \circ i}(z) \\ &= \mathbf{d}K(i(z)) \cdot T_z i(X_{H \circ i}(z)). \end{aligned}$$

Since  $K$  is arbitrary, we conclude that  $X_H(i(z)) = T_z i(X_{H \circ i}(z))$ . ■

If  $S \subset P$  is a submanifold of  $P$  and the inclusion  $i$  is Poisson, we say that  $S$  is a **Poisson submanifold** of  $P$ . Note that the only immersed Poisson submanifolds of a symplectic manifold are those whose range in  $P$  is open since for any (weak) symplectic manifold  $P$ , we have

$$T_z P = \{X_H(z) \mid H \in \mathcal{F}(U), \quad U \text{ open in } P\}.$$

Note that any Hamiltonian vector field must be tangent to a Poisson submanifold. Also note that the only Poisson submanifolds of a symplectic manifold  $P$  are its open sets.

**Symplectic Stratifications.** Now we come to an important result that states that every Poisson manifold is a union of symplectic manifolds, each of which is a Poisson submanifold.

**Definition 10.4.3.** *Let  $P$  be a Poisson manifold. We say that  $z_1, z_2 \in P$  are **on the same symplectic leaf** of  $P$  if there is a piecewise smooth curve in  $P$  joining  $z_1$  and  $z_2$ , each segment of which is a trajectory of a locally defined Hamiltonian vector field. This is clearly an equivalence relation and an equivalence class is called a **symplectic leaf**. The symplectic leaf containing the point  $z$  is denoted  $\Sigma_z$ .*

**Theorem 10.4.4 (Symplectic Stratification Theorem).** *Let  $P$  be a finite dimensional Poisson manifold. Then  $P$  is the disjoint union of its symplectic leaves. Each symplectic leaf in  $P$  is an injectively immersed Poisson submanifold and the induced Poisson structure on the leaf is symplectic. The dimension of the leaf through a point  $z$  equals the rank of the Poisson structure at that point and the tangent space to the leaf at  $z$  equals*

$$B^\#(z)(T_z^*P) = \{X_H(z) \mid H \in \mathcal{F}(U), U \text{ open in } P\}.$$

The Poisson bracket on  $P$  can be alternatively described as follows.

*To evaluate the Poisson bracket of  $F$  and  $G$  at  $z \in P$ , restrict  $F$  and  $G$  to the symplectic leaf  $\Sigma$  through  $z$ , take their bracket on  $\Sigma$  (in the sense of brackets on a symplectic manifold), and evaluate at  $z$ .*

Also note that since the Casimir functions have differentials that annihilate the characteristic field, they are constant on symplectic leaves.

To get a feeling for the geometric content of the symplectic stratification theorem, let us first prove it under the assumption that the characteristic field is a smooth vector subbundle of  $TP$  which is the case considered originally by Lie [1890]. In finite dimensions, this is guaranteed if the rank of the Poisson structure is constant. Jacobi's identity shows that the characteristic field is involutive and thus by the Frobenius theorem, it is integrable. Therefore,  $P$  is foliated by injectively immersed submanifolds whose tangent space at any point coincides with the subspace of all Hamiltonian vector fields evaluated at  $z$ . Thus, each such leaf  $\Sigma$  is an immersed Poisson submanifold of  $P$ . Define the two-form  $\Omega$  on  $\Sigma$  by

$$\Omega(z)(X_F(z), X_G(z)) = \{F, G\}(z)$$

for any functions  $F, G$  defined on a neighborhood of  $z$  in  $P$ . Note that  $\Omega$  is closed by the Jacobi identity (Exercise 5.5-1). Also, if

$$0 = \Omega(z)(X_F(z), X_G(z)) = \mathbf{d}F(z) \cdot X_G(z)$$

for all locally defined  $G$ , then

$$\mathbf{d}F(z)|_{T_z\Sigma} = \mathbf{d}(F \cdot i)(z)$$

by the Hahn–Banach theorem. Therefore,

$$0 = X_{F \circ i}(z) = T_z i(X_F(z)) = X_F(z),$$

since  $\Sigma$  is a Poisson submanifold of  $P$  and the inclusion  $i : \Sigma \rightarrow P$  is a Poisson map, thus showing that  $\Omega$  is weakly nondegenerate and thereby proving the theorem for the constant rank case.

The general case, proved by Kirillov [1976a], is more subtle since for differentiable distributions which are not subbundles, integrability and involutivity are not equivalent. To prove this case, we proceed in a series of technical propositions.<sup>5</sup>

**Proposition 10.4.5.** *Let  $P$  be a finite dimensional Poisson manifold with  $B_z^\sharp : T_z^*P \rightarrow T_zP$  the Poisson tensor. Take  $z \in P$  and functions  $f_1, \dots, f_k$  defined on  $P$  such that  $\{B_z^\sharp df_j\}_{1 \leq j \leq k}$  is a basis of the range of  $B_z^\sharp$ . Let  $\Phi_{j,t}$  be the local flow defined in a neighborhood of  $z$  generated by the Hamiltonian vector field  $X_{f_j} = B^\sharp df_j$ . Let*

$$\Psi_{f_1, \dots, f_k}^z(t_1, \dots, t_k) = (\Phi_{1,t_1} \circ \dots \circ \Phi_{k,t_k})(z)$$

for small enough  $t_1, \dots, t_k$ . Then:

- (i) *There is an open neighborhood  $U_\delta$  of  $0 \in \mathbb{R}^k$  such that:*

$$\Psi_{f_1, \dots, f_k}^z : U_\delta \rightarrow P$$

*is an embedding.*

- (ii) *The ranges of  $(T\Psi_{f_1, \dots, f_k}^z)(t)$  and  $B_{\Psi_{f_1, \dots, f_k}^z(t)}^\sharp$  are equal for  $t \in U_\delta$ .*

- (iii)  $\Psi_{f_1, \dots, f_k}^z(U_\delta) \subset \Sigma_z$ .

- (iv) *If*

$$\Psi_{g_1, \dots, g_k}^y : U_\eta \rightarrow P$$

*is another map constructed as above and  $y \in \Psi_{f_1, \dots, f_k}^z(U_\delta)$ , then there is an open subset,  $U_\epsilon \subset U_\eta$ , such that  $\Psi_{g_1, \dots, g_k}^y$  is a diffeomorphism from  $U_\epsilon$  to an open subset in  $\Psi_{f_1, \dots, f_k}^z(U_\delta)$ .*

**Proof.** (i) The smoothness of  $\Psi_{f_1, \dots, f_k}^z$  follows from the smoothness of  $\Phi_{j,t}$  in both the flow parameter and manifold variables. Then

$$T_0 \Psi_{f_1, \dots, f_k}^z(\partial/\partial t_j) = X_{f_j}(z) = B_z^\sharp df_j,$$

which shows that  $T_0 \Psi_{f_1, \dots, f_k}^z$  is injective. It follows that  $\Psi_{f_1, \dots, f_k}^z$  is an embedding on a sufficiently small neighborhood of 0, say  $U_\delta$ . Notice also that the ranges of  $T_0 \Psi_{f_1, \dots, f_k}^z$  and of  $B_z^\sharp$  coincide.

- (ii) From 10.5.2 we recall that for any invertible Poisson map  $\Phi$  on  $P$ , we have  $T\Phi \cdot X_f = X_{f \circ \Phi^{-1}} \circ \Phi$  and from 10.5.1 we know that the Hamiltonian flows are Poisson maps. Therefore, if  $t = (t_1, \dots, t_k)$ ,

Is 10.5.2  
a Theorem,  
etc.?

$$\begin{aligned} T_t \Psi_{f_1, \dots, f_k}^z(\partial/\partial t_j) &= (T\Phi_{1,t_1} \circ \dots \circ T\Phi_{j-1,t_{j-1}} \circ X_{f_j} \circ \Phi_{j+1,t_{j+1}} \circ \dots \circ \Phi_{k,t_k})(z) \\ &= (X_{h_j} \circ \Psi_{f_1, \dots, f_k}^z)(t), \end{aligned}$$

<sup>5</sup>This proof was kindly supplied by O. Popp

where

$$h_j = f_j \circ (\Phi_{1,t_1} \circ \dots \circ \Phi_{j-1,t_{j-1}})^{-1}.$$

This shows that

$$\text{range } T_t \Psi_{f_1, \dots, f_k}^x \subset \text{range } B_{\Psi_{f_1, \dots, f_k}^x}^\sharp(t)$$

if  $t \in U_\delta$ . Since  $B^\sharp$  is invariant under Hamiltonian flows, it follows that

$$\dim \text{range } B_{\Psi_{f_1, \dots, f_k}^z}^\sharp(t) = \dim \text{range } B_z^\sharp.$$

This last equality, the previous inclusion, and the last remark in the proof of (i) above conclude (ii).

(iii) This is obvious since  $\Psi_{f_1, \dots, f_k}^z$  is built from piecewise Hamiltonian curves starting from  $z$ .

(iv) Note that  $X_g(z) \in \text{range } B_z^\sharp$  for any  $z \in P$  and any smooth function  $g$ . Using (ii), we see that  $X_g$  is tangent to the image of  $\Psi_{f_1, \dots, f_k}^z$ . Therefore, the integral curves of  $X_g$  remain tangent to  $\Psi_{f_1, \dots, f_k}^z(U_\delta)$  if they start from that set. To get  $\Psi_{g_1, \dots, g_k}^y$  we just have to find Hamiltonian curves which start from  $y$ . Therefore, we can restrict ourselves to the submanifold  $\Psi_{f_1, \dots, f_k}^z(U_\delta)$  when computing the flows along the Hamiltonian vector fields  $X_{g_j}$ ; therefore we can consider that the image of  $\Psi_{g_1, \dots, g_k}^y$  is in  $\Psi_{f_1, \dots, f_k}^z(U_\delta)$ ; now the derivative at  $0 \in \mathbb{R}^k$  of  $\Psi_{g_1, \dots, g_k}^y$  is an isomorphism to the tangent space of  $\Psi_{f_1, \dots, f_k}^z(U_\delta)$  at  $y$  (that is,  $\text{range } B_y^\sharp$ ), using (ii) above. Thus, the existence of the neighborhood  $U_\epsilon$  follows from the inverse function theorem. ■

**Proposition 10.4.6.** *Let  $P$  be a Poisson manifold and  $B$  its Poisson tensor. Then for each symplectic leaf  $\Sigma \subset P$ , the family of charts satisfying (i) in the previous proposition, namely,*

$$\{\Psi_{f_1, \dots, f_k}^z \mid z \in \Sigma, \{B_z^\sharp df_j\}_{1 \leq j \leq k} \text{ a basis for } \text{range } B_z^\sharp\},$$

*gives  $\Sigma$  a structure of differentiable manifold such that the inclusion is an immersion. Then  $T_z \Sigma = \text{range } B_z^\sharp$  (so  $\dim \Sigma = \text{rank } B_z^\sharp$ ), for all  $z \in \Sigma$ . Moreover,  $\Sigma$  has a unique symplectic structure such that the inclusion is a Poisson map.*

**Proof.** Let  $w \in \Psi_{f_1, \dots, f_k}^z(U_\delta) \cap \Psi_{g_1, \dots, g_k}^y(U_\epsilon)$  and consider  $\Psi_{h_1, \dots, h_k}^w : U_\gamma \rightarrow P$ . Using (iv) in the proposition above, we can choose  $U_\gamma$  small enough so that

$$\Psi_{h_1, \dots, h_k}^w(U_\gamma) \subset \Psi_{f_1, \dots, f_k}^z(U_\delta) \cap \Psi_{g_1, \dots, g_k}^y(U_\epsilon)$$

is a diffeomorphic embedding in both  $\Psi_{f_1, \dots, f_k}^z(U_\delta)$  and  $\Psi_{g_1, \dots, g_k}^y(U_\epsilon)$ . This shows that the transition maps for the given charts are diffeomorphisms and so define the structure of a differentiable manifold on  $\Sigma$ . The fact that the inclusion is an immersion follows from (i) of the above proposition. We get the tangent space of  $\Sigma$  using (i), (ii) of the previous proposition; then the equality of dimensions follows.

It follows from the definition of an immersed Poisson submanifold that  $\Sigma$  is such a submanifold of  $P$ . Thus, if  $i : \Sigma \rightarrow P$  is the inclusion,

$$\{f \circ i, g \circ i\}_\Sigma = \{f, g\} \circ i.$$

Hence if  $\{f \circ i, g \circ i\}_\Sigma(z) = 0$  for all functions  $g$  then  $\{f, g\}(z) = 0$  for all  $g$ , that is,  $(X_g f)(z) = 0$  for all  $g$ . This implies that  $df|_{T_z \Sigma} = 0$  since the vectors  $X_g(z)$  span  $T_z \Sigma$ . Therefore,  $i^* \mathbf{d}f = \mathbf{d}(f \circ i) = 0$ , which shows that the Poisson tensor on  $\Sigma$  is nondegenerate and thus  $\Sigma$  is a symplectic manifold. This proves the proposition and also completes the proof of the symplectic stratification theorem. ■

There is another proof of the above theorem (using the same idea as for the Darboux coordinates) in Weinstein [1983] (see Libermann and Marle [1987] also.) The proof given above is along the Frobenius integrability idea. Actually it can be used to produce a proof of the generalized Frobenius theorem.

**Theorem 10.4.7 (Singular Frobenius Theorem).** *Let  $D$  be a distribution of subspaces of the tangent bundle of a finite dimensional manifold  $M$ , that is,  $D_x \subset T_x M$  as  $x$  varies in  $M$ . Suppose it is smooth in the sense that for each  $x$  there are smooth vector fields  $X_i$  in  $D$  such that  $X_i(x)$  give a basis of  $D_x$ . Then  $D$  is integrable, that is, for each  $x \in M$  there is an immersed submanifold  $\Sigma_x \subset M$  with  $T_x \Sigma_x = D_x$  if and only if the distribution  $D$  is invariant under the (local) flows along vector fields with values in  $D$ .*

**Proof.** The only if part follows easily. For the if part we remark that the proof of the theorem above can be reproduced here replacing range  $B_z^\sharp$  by  $D_x$  and the Hamiltonian vector fields with vector fields in  $D$ . The crucial property needed to prove (ii) in the above proposition (i.e. Hamiltonian fields remain Hamiltonian under Hamiltonian flows) is replaced by the invariance of  $D$  given in the hypothesis. ■

#### Remarks.

1. The conclusion of the above theorem is the same as the Frobenius integrability theorem but it is not supposed that the dimension of  $D_x$  is constant.

..... 11 March 1998—16h24 .....

**2.** Analogous to the symplectic leaves of a Poisson manifold, we can define the *maximal integral manifolds* of the integrable distribution  $D$  using curves along vector fields in  $D$  instead of Hamiltonian vector fields. They are also immersed submanifolds in  $M$ .

**3.** The condition that (local) flows of the vector fields with values in  $D$  leave  $D$  invariant implies the involution property of  $D$ , that is,  $[X, Y]$  is a vector field with values in  $D$  if both  $X$  and  $Y$  are vector fields with values in  $D$  (use (4.3.7)). But the involution property alone is not enough to guarantee that  $D$  is integrable (if the dimension of  $D$  is not constant).

**4.** This generalization of the Frobenius integrability theorem is due to Hermann [1964], Stefan [1974], Sussman [1973], and it has proved quite useful in control theory; see also Libermann and Marle [1987]. ♦

## Examples

**(a)** Let  $P = \mathbb{R}^3$  with the rigid body bracket. Then the symplectic leaves are spheres centered at the origin. The single point at the origin is the singular leaf in the sense that the Poisson structure has rank zero there. As we shall see later, it is true more generally that the symplectic leaves in  $\mathfrak{g}^*$  with the Lie–Poisson bracket are the coadjoint orbits. ♦

**(b)** Symplectic leaves need not be submanifolds and *one cannot conclude that if all the Casimir functions are constants then the Poisson structure is nondegenerate*. For example, consider  $\mathbb{T}^3$  with a codimension 1 foliation with dense leaves, such as obtained by taking the leaves to be the product of  $\mathbb{T}^1$  with a leaf of the irrational flow on  $\mathbb{T}^2$ . Put the usual area element on these leaves and define a Poisson structure on  $\mathbb{T}^3$  by declaring these to be the symplectic leaves. Any Casimir function is constant, yet the Poisson structure is degenerate. ♦

**Poisson–Darboux Theorem.** Related to the stratification theorem is an analogue of Darboux’ theorem. To state it, first recall from Exercise 10.5-2 that we define the product Poisson structure on  $P_1 \times P_2$  where  $P_1, P_2$  are Poisson manifolds by the requirements that the projections  $\pi_1 : P_1 \times P_2 \rightarrow P_1$  and  $\pi_2 : P_1 \times P_2 \rightarrow P_2$  are Poisson mappings, and  $\pi_1^*(\mathcal{F}(P_1))$  and  $\pi_2^*(\mathcal{F}(P_2))$  are commuting subalgebras of  $\mathcal{F}(P_1 \times P_2)$ . In terms of coordinates, if bracket relations  $\{z^I, z^J\} = B^{IJ}(z)$  and  $\{w^I, w^J\} = C^{IJ}(w)$  are given on  $P_1$  and  $P_2$ , respectively, then these define a bracket on functions of  $z^I$  and  $w^J$  when augmented by the relations  $\{z^I, w^J\} = 0$ .

**Theorem 10.4.8 (Lie–Weinstein).** *Let  $z_0$  be a point in a Poisson manifold  $P$ . There is a neighborhood  $U$  of  $z_0$  in  $P$  and an isomorphism  $\varphi = \varphi_S \times \varphi_N : U \rightarrow S \times N$ , where  $S$  is symplectic,  $N$  is Poisson, and the rank of  $N$  at  $\varphi_N(z_0)$  is zero. The factors  $S$  and  $N$  are unique up to local isomorphism. Moreover, if the rank of the Poisson manifold is constant near  $z_0$ ,*

..... 11 March 1998—16h24 .....



there are coordinates  $(q^1, \dots, q^k, p_1, \dots, p_k, y^1, \dots, y^l)$  near  $x_0$  satisfying the canonical bracket relations

$$\{q^i, q^j\} = \{p_i, p_j\} = \{q^i, y^j\} = \{p_i, y^j\} = 0, \{q^i, p_j\} = \delta_j^i.$$

In this theorem, the manifold  $S$  can be taken to be the symplectic leaf of  $P$  through  $z_0$  and  $N$  is, locally, any submanifold of  $P$ , transverse to  $S$ , and such that  $S \cap N = \{z_0\}$ . In many cases the transverse structure on  $N$  is of Lie–Poisson type. For the proof of this theorem and related results, see Weinstein [1983b]; the second part of the theorem is due to Lie [1890]. For the main examples in this book, we shall not require a detailed local analysis of their Poisson structure, so we shall forego a more detailed study of the local structure of Poisson manifolds.

### Exercises

- ◇ **Exercise 10.4-1.** If  $H \in \mathcal{F}(P)$ , where  $P$  is a Poisson manifold, show that the flow  $\varphi_t$  of  $X_H$  preserves the symplectic leaves of  $P$ .
- ◇ **Exercise 10.4-2.** Let  $(P, \{ , \})$  be a Poisson manifold with Poisson tensor  $B \in \Omega_2(P)$ . Let

$$B^\sharp : T^*P \rightarrow TP, \quad B^\sharp(\mathbf{d}H) = X_H,$$

be the induced bundle map. We shall denote by the same symbol  $B^\sharp : \Omega^1(P) \rightarrow \mathfrak{X}(P)$  the induced map on the sections. The definitions introduced in §10.3 and §10.6 read

$$B(\mathbf{d}F, \mathbf{d}H) = \langle \mathbf{d}F, B^\sharp(\mathbf{d}H) \rangle = \{F, H\}.$$

Define  $\alpha^\sharp := B^\sharp(\alpha)$ . Define for any  $\alpha, \beta \in \Omega^1(P)$ ,

$$\{\alpha, \beta\} = -\mathcal{L}_{\alpha^\sharp}\beta + \mathcal{L}_{\beta^\sharp}\alpha - \mathbf{d}(B(\alpha, \beta)).$$

- (a) Show that if the Poisson bracket on  $P$  is induced by a symplectic form  $\Omega$ , that is, if  $B^\sharp = \Omega^\sharp$ , then

$$B(\alpha, \beta) = \Omega(\alpha^\sharp, \beta^\sharp).$$

- (b) Show that, for any  $F, G \in \mathcal{F}(P)$ , we have

$$\{F\alpha, G\beta\} = FG\{\alpha, \beta\} - F\alpha^\sharp[G]\beta + G\beta^\sharp[F]\alpha.$$

- (c) Show that, for any  $F, G \in \mathcal{F}(P)$  we have

$$\mathbf{d}\{F, G\} = \{\mathbf{d}F, \mathbf{d}G\}.$$

- (d) Show that, if  $\alpha, \beta \in \Omega^1(P)$  are closed, then,  $\{\alpha, \beta\} = \mathbf{d}(B(\alpha, \beta))$ .

- (e) Use  $\mathcal{L}_{X_H}B = 0$  to show that  $\{\alpha, \beta\}^\sharp = -[\alpha^\sharp, \beta^\sharp]$ .
- (f) Show that  $(\Omega^1(P), \{ , \})$  is a Lie algebra; that is, prove Jacobi's identity.

◇ **Exercise 10.4-3 (Weinstein [1983]).** Let  $P$  be a manifold and  $X, Y$  be two linearly independent commuting vector fields. Show that

$$\{F, K\} = X[F]Y[K] - Y[F]X[K]$$

defines a Poisson bracket on  $P$ . Show that

$$X_H = Y[H]X - X[H]Y.$$

Show that the symplectic leaves are two-dimensional and that their tangent spaces are spanned by  $X$  and  $Y$ . Show how to get Example (b) preceding 10.4.8 from this construction.

## 10.5 Quotients of Poisson Manifolds

Here we shall give the simplest version of a general construction of Poisson manifolds based on symmetry. This construction represents the first steps in a general procedure called *reduction*.

**Poisson Reduction Theorem.** Suppose that  $G$  is a Lie group that acts on a Poisson manifold and that each map  $\Phi_g : P \rightarrow P$  is a Poisson map. Let us also suppose that the action is free and proper, so that the quotient space  $P/G$  is a smooth manifold and the projection  $\pi : P \rightarrow P/G$  is a submersion (see the discussion of this point in §9.3).

**Theorem 10.5.1.** *Under these hypotheses, there is a unique Poisson structure on  $P/G$  such that  $\pi$  is a Poisson map.*

**Proof.** Let us first assume  $P/G$  is Poisson and shows uniqueness. The condition that  $\pi$  be Poisson is that for two functions  $f, k : P/G \rightarrow \mathbb{R}$ ,

$$\{f, k\} \circ \pi = \{f \circ \pi, k \circ \pi\}, \quad (10.5.1)$$

where the brackets are on  $P/G$  and  $P$ , respectively. The function  $\bar{f} = f \circ \pi$  is the unique  $G$ -invariant function that projects to  $f$ . In other words, if  $[z] \in P/G$  is an equivalence class, whereby  $g_1 \cdot z$  and  $g_2 \cdot z$  are equivalent, we let  $\bar{f}(g \cdot z) = f([z])$  for all  $g \in G$ . Obviously, this defines  $\bar{f}$  unambiguously, so that  $\bar{f} = f \circ \pi$ . We can also characterize this as saying that  $\bar{f}$  assigns the value  $f([z])$  to the whole orbit  $G \cdot z$ . We can write (10.5.1) as

$$\{f, k\} \circ \pi = \{\bar{f}, \bar{k}\}.$$

..... 11 March 1998—16h24 .....

Since  $\pi$  is onto, this determines  $\{f, k\}$  uniquely.

We can also use (10.5.1) to *define*  $\{f, k\}$ . First, note that

$$\begin{aligned}\{\bar{f}, \bar{k}\}(g \cdot z) &= (\{\bar{f}, \bar{k}\} \circ \Phi_g)(z) \\ &= \{\bar{f} \circ \Phi_g, \bar{k} \circ \Phi_g\}(z) \\ &= \{\bar{f}, \bar{k}\}(z),\end{aligned}$$

since  $\Phi_g$  is Poisson and since  $\bar{f}$  and  $\bar{k}$  are constant on orbits. Thus,  $\{\bar{f}, \bar{k}\}$  is constant on orbits too, and so it defines  $\{f, k\}$  uniquely.

It remains to show that  $\{f, k\}$  so defined satisfies the properties of a Poisson structure. However, these all follow from their counterparts on  $P$ . For example, if we write Jacobi's identity on  $P$ , namely

$$0 = \{\{\bar{f}, \bar{k}\}, \bar{l}\} + \{\{\bar{l}, \bar{f}\}, \bar{k}\} + \{\{\bar{k}, \bar{l}\}, \bar{f}\},$$

it gives, by construction,

$$\begin{aligned}0 &= \{\{f, k\} \circ \pi, l \circ \pi\} + \{\{l, f\} \circ \pi, k \circ \pi\} + \{\{k, l\} \circ \pi, f \circ \pi\} \\ &= \{\{f, k\}, l\} \circ \pi + \{\{l, f\}, k\} \circ \pi + \{\{k, l\}, f\} \circ \pi\end{aligned}$$

and thus by surjectivity of  $\pi$ , Jacobi's identity holds on  $P/G$ . ■

This construction is just one of many that produce new Poisson and symplectic manifolds from old ones. We refer to Marsden and Ratiu [1986] and Vaisman [1996] for generalizations of the construction here.

**Reduction of Dynamics.** If  $H$  is a  $G$ -invariant Hamiltonian on  $P$ , it defines a corresponding function  $h$  on  $P/G$  such that  $H = h \circ \pi$ . Since  $\pi$  is a Poisson map, it transforms  $X_H$  on  $P$  to  $X_h$  on  $P/G$ ; that is,  $T\pi \circ X_H = X_h \circ \pi$ , or  $X_H$  and  $X_h$  are  $\pi$ -related. We say that the Hamiltonian system  $X_H$  on  $P$  **reduces** to that on  $P/G$ .

As we shall see in the next chapter,  $G$ -invariance of  $H$  may be associated with a conserved quantity  $J : P \rightarrow \mathbb{R}$ . If it is also  $G$ -invariant, the corresponding function  $j$  on  $P/G$  is conserved for  $X_h$  since

$$\{h, j\} \circ \pi = \{H, J\} = 0$$

and so  $\{h, j\} = 0$ .

**Example.** Consider the differential equations on  $\mathbb{C}^2$  given by

$$\left. \begin{aligned}\dot{z}_1 &= -i\omega_1 z_1 + i\epsilon p \bar{z}_2 + iz_1(s_{11}|z_1|^2 + s_{12}|z_2|^2), \\ \dot{z}_2 &= -i\omega_2 z_2 + i\epsilon q \bar{z}_1 - iz_2(s_{21}|z_1|^2 + s_{22}|z_2|^2).\end{aligned}\right\} \quad (10.5.2)$$

Use the standard Hamiltonian structure obtained by taking the real and imaginary parts of  $z_i$  as conjugate variables. For example, we write

..... 11 March 1998—16h24 .....

$z_1 = q_1 + ip_1$  and require  $\dot{q}_1 = \partial H / \partial p_1$  and  $\dot{p}_1 = -\partial H / \partial q_1$ . Recall from Chapter 5 that a useful trick in this regard, that enables one to work in complex notation, is to write Hamilton's equations as  $\dot{z}_k = -2i\partial H / \partial \bar{z}_k$ . Using this, one readily finds that (see Exercise 5.4-3): *The system (10.5.2) is Hamiltonian if and only if  $s_{12} = -s_{21}$  and  $p = q$ .* In this case we can choose

$$H(z_1, z_2) = \frac{1}{2}(\omega_2 |z_2|^2 + \omega_1 |z_1|^2) - \epsilon p \operatorname{Re}(z_1 z_2) - \frac{s_{11}}{4} |z_1|^4 - \frac{s_{12}}{2} |z_1 z_2|^2 + \frac{s_{22}}{4} |z_2|^4. \quad (10.5.3)$$

Note that for equation (10.5.2) with  $\epsilon = 0$  there are two separate  $S^1$  actions acting on  $z_1$  and  $z_2$  independently; corresponding conserved quantities are  $|z_1|^2$  and  $|z_2|^2$ . However, for  $\epsilon \neq 0$ , the symmetry action is

$$(z_1, z_2) \mapsto (e^{i\theta} z_1, e^{-i\theta} z_2) \quad (10.5.4)$$

with the conserved quantity (Exercise 5.5-3)

$$J(z_1, z_2) = \frac{1}{2}(|z_1|^2 - |z_2|^2). \quad (10.5.5)$$

Let  $\phi = \frac{\pi}{2} - \theta_1 - \theta_2$ , where  $z_1 = r_1 \exp(i\theta_1)$ ,  $z_2 = r_2 \exp(i\theta_2)$ . We know that the Hamiltonian structure for (10.5.2) on  $\mathbb{C}^2$  described above induces one on  $\mathbb{C}^2/S^1$  (exclude points where  $r_1$  or  $r_2$  vanishes), and that the two integrals (energy and the conserved quantity) descend to the quotient space, as does the Poisson bracket. The quotient space  $\mathbb{C}^2/S^1$  is parametrized by  $(r_1, r_2, \phi)$  and  $H$  and  $J$  can be dropped to the quotient. Concretely, the process of dropping to the quotient is very simple: if  $F(z_1, z_2) = F(r_1, \theta_1, r_2, \theta_2)$  is  $S^1$  invariant, then it can be written (uniquely) as a function  $f$  of  $(r_1, r_2, \phi)$ .

By Theorem 10.5.1, one can also drop the Poisson bracket to the quotient. Consequently, the equations in  $(r_1, r_2, \phi)$  can be cast in Hamiltonian form  $\dot{f} = \{f, h\}$  for the induced Poisson bracket. This bracket is obtained by using the chain rule to relate the complex variables and the polar coordinates. One finds that

$$\begin{aligned} \{f, k\}(r_1, r_2, \phi) \\ = -\frac{1}{r_1} \left( \frac{\partial f}{\partial r_1} \frac{\partial k}{\partial \phi} - \frac{\partial f}{\partial \phi} \frac{\partial k}{\partial r_1} \right) - \frac{1}{r_2} \left( \frac{\partial f}{\partial r_2} \frac{\partial k}{\partial \phi} - \frac{\partial f}{\partial \phi} \frac{\partial k}{\partial r_2} \right). \end{aligned} \quad (10.5.6)$$

The (non-canonical) Poisson bracket (10.5.6) is, of course, the reduction of the original *canonical* Poisson bracket on the space of  $q$  and  $p$  variables, written in the new polar coordinate variables. Theorem 10.5.1 shows that Jacobi's identity is automatic for this reduced bracket. (See Knobloch, Mahalov, and Marsden [1994] for further examples of this type.) ♦

As we shall see in Chapter 13, a key example of the Poisson reduction given in 10.5.1 is when  $P = T^*G$  and  $G$  acts on itself by left translations. Then  $P/G \cong \mathfrak{g}^*$  and the reduced Poisson bracket is none other than the Lie–Poisson bracket!

### Exercises

- ◇ **Exercise 10.5-1.** Let  $\mathbb{R}^3$  be equipped with the rigid body bracket and let  $G = S^1$  act on  $P = \mathbb{R}^3 \setminus (z\text{-axis})$  by rotation about the  $z$ -axis. Compute the induced bracket on  $P/G$ .
- ◇ **Exercise 10.5-2.** Compute explicitly the reduced Hamiltonian  $h$  in the example in the text and verify directly that the equations for  $\dot{r}_1, \dot{r}_2, \dot{\phi}$  are Hamiltonian on  $\mathbb{C}^2$  with Hamiltonian  $h$ . Also check that the function  $j$  induced by  $J$  is a constant of the motion.

## 10.6 The Schouten Bracket

The goal of this subsection is to express the Jacobi identity for a Poisson structure in geometric terms analogous to  $\mathbf{d}\Omega$  for symplectic structures. This will be done in terms of a bracket defined on contravariant antisymmetric tensors generalizing the Lie bracket of vector fields (see, for example, Schouten [1940], Nijenhuis [1953], Lichnerowicz [1978], Olver [1984, 1986], Koszul [1985], Libermann and Marle [1987], Bhaskara and Viswanath [1988], Kosman–Schwarzbach and Magri [1990], Vaisman [1994], and references therein).

**Multivectors.** A *contravariant antisymmetric  $q$ -tensor* on a finite-dimensional vector space  $V$  is a  $q$ -linear map

$$A : V^* \times V^* \times \cdots \times V^* (q \text{ times}) \rightarrow \mathbb{R}$$

that is antisymmetric in each pair of arguments. The space of these tensors will be denoted by  $\Lambda_q(V)$ . Thus, each element  $\Lambda_q(V)$  is a finite linear combination of terms of the form  $v_1 \wedge \cdots \wedge v_q$ , called a  *$q$ -vector*, for  $v_1, \dots, v_q \in V$ . If  $V$  is an infinite-dimensional Banach space, we define  $\Lambda_q(V)$  to be the span of all elements of the form  $v_1 \wedge \cdots \wedge v_q$  with  $v_1, \dots, v_q \in V$ , where the exterior product is defined in the usual manner relative to a weakly non-degenerate pairing  $\langle, \rangle : V^* \times V \rightarrow \mathbb{R}$ . Thus,  $\Lambda_0(V) = \mathbb{R}$  and  $\Lambda_1(V) = V$ . If  $P$  is a smooth manifold, let

$$\Lambda_q(P) = \bigcup_{z \in P} \Lambda_q(T_z P),$$

a smooth vector bundle with fiber over  $z \in P$  equal to  $\Lambda_q(T_z P)$ . Let  $\Omega_q(P)$  denote the smooth sections of  $\Lambda_q(P)$ , that is, the elements of  $\Omega_q(P)$  are

..... 11 March 1998—16h24 .....

smooth contravariant antisymmetric  $q$ -tensor fields on  $P$ . Let  $\Omega_*(P)$  be the direct sum of the spaces  $\Omega_q(P)$ , where  $\Omega_0(P) = \mathcal{F}(P)$ . Note that

$$\Omega_q(P) = 0 \quad \text{for } q > \dim(P),$$

and that

$$\Omega_1(P) = \mathfrak{X}(P).$$

If  $X_1, \dots, X_q \in \mathfrak{X}(P)$ ,  $X_1 \wedge \dots \wedge X_q$  is called a ***q-vector field***, or a ***multivector field***.

On the manifold  $P$ , consider a  $(q+p)$ -form  $\alpha$  and a contravariant antisymmetric  $q$ -tensor  $A$ . The ***interior product***  $\mathbf{i}_A \alpha$  of  $A$  with  $\alpha$  is defined as follows. If  $q = 0$ , so  $A \in \mathbb{R}$ , let  $\mathbf{i}_A \alpha = A\alpha$ . If  $q \geq 1$  and if  $A = v_1 \wedge \dots \wedge v_q$ , where  $v_i \in T_z P$ ,  $i = 1, \dots, q$ , define  $\mathbf{i}_A \alpha \in \Omega^p(P)$  by

$$(\mathbf{i}_A \alpha)(v_{q+1}, \dots, v_{q+p}) = \alpha(v_1, \dots, v_{q+p}) \quad (10.6.1)$$

for arbitrary  $v_{q+1}, \dots, v_{q+p} \in T_z P$ . One checks that the definition does not depend on the representation of  $A$  as a  $q$ -vector, so  $\mathbf{i}_A \alpha$  is well defined on  $\Lambda_q(P)$  by linear extension. In local coordinates, for finite-dimensional  $P$ ,

$$(\mathbf{i}_A \alpha)_{i_{q+1} \dots i_{q+p}} = A^{i_1 \dots i_q} \alpha_{i_1 \dots i_{q+p}}, \quad (10.6.2)$$

where all components are nonstrict. If  $P$  is finite dimensional and  $p = 0$ , (10.6.1) defines an isomorphism of  $\Omega_q(P)$  with  $\Omega^q(P)$ . If  $P$  is a Banach manifold, (10.6.1) defines a weakly nondegenerate pairing of  $\Omega_q(P)$  with  $\Omega^q(P)$ . If  $A \in \Omega_q(P)$ ,  $q$  is called the ***degree*** of  $A$  and is denoted by  $\deg A$ . One checks that

$$\mathbf{i}_{A \wedge B} \alpha = \mathbf{i}_B \mathbf{i}_A \alpha. \quad (10.6.3)$$

The Lie derivative  $\mathcal{L}_X$  is a derivation relative to  $\wedge$ , that is,

$$\mathcal{L}_X(A \wedge B) = (\mathcal{L}_X A) \wedge B + A \wedge (\mathcal{L}_X B)$$

for any  $A, B \in \Omega_*(P)$ .

**The Schouten Bracket.** The next theorem produces an interesting bracket on multivectors.

**Theorem 10.6.1 (Schouten Bracket Theorem).** *There is a unique bilinear operation  $[\cdot, \cdot] : \Omega_*(P) \times \Omega_*(P) \rightarrow \Omega_*(P)$  natural with respect to restriction to open sets, called the **Schouten bracket**, that satisfies the following properties:*

- (i) *it is a **biderivation of degree**  $-1$ , that is, it is bilinear,*

$$\deg[A, B] = \deg A + \deg B - 1, \quad (10.6.4)$$

*and for  $A, B, C \in \Omega_*(P)$ ,*

$$[A, B \wedge C] = [A, B] \wedge C + (-1)^{(\deg A + 1) \deg B} B \wedge [A, C]; \quad (10.6.5)$$

- (ii) *it is determined on  $\mathcal{F}(P)$  and  $\mathfrak{X}(P)$  by*
- (a)  $[F, G] = 0$ , for all  $F, G \in \mathcal{F}(P)$ ;
  - (b)  $[X, F] = X[F]$ , for all  $F \in \mathcal{F}(P)$ ,  $X \in \mathfrak{X}(P)$ ;
  - (c)  $[X, Y]$  for all  $X, Y \in \mathfrak{X}(P)$  is the usual Jacobi–Lie bracket of vector fields; and
- (iii)  $[A, B] = (-1)^{\deg A \deg B} [B, A]$ .

In addition, the Schouten bracket satisfies the **graded Jacobi identity**

$$\begin{aligned} (-1)^{\deg A \deg C} [[A, B], C] + (-1)^{\deg B \deg A} [[B, C], A] \\ + (-1)^{\deg C \deg B} [[C, A], B] = 0. \end{aligned} \quad (10.6.6)$$

**Proof.** The proof proceeds in standard fashion and is similar to that characterizing the exterior or Lie derivative by its properties, see (Abraham, Marsden, and Ratiu [1988]): on functions and vector fields it is given by (ii); then (i) and linear extension determine it on any skew-symmetric contravariant tensor in the second variable and a function and vector field in the first; (iii) tells how to switch such variables and finally (i) again defines it on any pair of skew-symmetric contravariant tensors. The operation so defined satisfies (i), (ii), and (iii) by construction. Uniqueness is a consequence of the fact that the skew-symmetric contravariant tensors are generated as an exterior algebra locally by functions and vector fields and (ii). The graded Jacobi identity is verified on an arbitrary triple of  $q$ -,  $p$ -, and  $r$ -vectors using (i), (ii), and (iii) and then invoking trilinearity of the identity. ■

**Properties.** The following formulae are useful in computing with the Schouten bracket. If  $X \in \mathfrak{X}(P)$  and  $A \in \Omega_p(P)$ , induction on the degree of  $A$  and the use of property (i) show that

$$[X, A] = \mathcal{L}_X A. \quad (10.6.7)$$

An immediate consequence of this formula and the graded Jacobi identity is the *derivation property of the Lie derivative relative to the Schouten bracket*, that is,

$$\mathcal{L}_X [A, B] = [\mathcal{L}_X A, B] + [A, \mathcal{L}_X B], \quad (10.6.8)$$

for  $A \in \Omega_p(P)$ ,  $B \in \Omega_q(P)$ , and  $X \in \mathfrak{X}(P)$ . Using induction on the number of vector fields, (10.6.7), and the properties in Theorem 10.8.1, one can prove that

$$[X_1 \wedge \cdots \wedge X_r, A] = \sum_{i=1}^r (-1)^{i+1} X_1 \wedge \cdots \wedge \check{X}_i \wedge \cdots \wedge X_r \wedge (\mathcal{L}_{X_i} A), \quad (10.6.9)$$

where  $X_1, \dots, X_r \in \mathfrak{X}(P)$  and  $\check{X}_i$  means that  $X_i$  has been omitted. The last formula plus linear extension can be taken as the definition of the Schouten bracket and one can deduce Theorem 10.8.1 from it; see Vaisman [1994] for this approach. If  $A = Y_1 \wedge \dots \wedge Y_s$  for  $Y_1, \dots, Y_s \in \mathfrak{X}(P)$ , the formula above plus the derivation property of the Lie derivative give

$$\begin{aligned} [X_1 \wedge \dots \wedge X_r, Y_1 \wedge \dots \wedge Y_s] \\ = (-1)^{r+1} \sum_{i=1}^r \sum_{j=1}^s (-1)^{i+j} [X_i, Y_j] \wedge X_1 \wedge \dots \wedge \check{X}_i \wedge \dots \\ \wedge X_r \wedge Y_1 \wedge \dots \wedge \check{Y}_j \wedge \dots \wedge Y_s. \end{aligned} \quad (10.6.10)$$

Finally, if  $A \in \Omega_p(P)$ ,  $B \in \Omega_q(P)$ , and  $\alpha \in \Omega^{p+q-1}(P)$ , the formula

$$\mathbf{i}_{[A,B]}\alpha = (-1)^{q(p+1)} \mathbf{i}_A \mathbf{d} \mathbf{i}_B \alpha + (-1)^p \mathbf{i}_B \mathbf{d} \mathbf{i}_A \alpha - \mathbf{i}_B \mathbf{i}_A \mathbf{d} \alpha \quad (10.6.11)$$

(which is a direct consequence of (10.6.10) and Cartan's formula for  $\mathbf{d}\alpha$ ) can be taken as the definition of  $[A, B] \in \Omega_{p+q-1}(P)$ ; this is the approach taken originally in Nijenhuis [1955].

**Coordinate Formulas.** In local coordinates, denoting  $\partial/\partial z^i = \partial_i$ , the formulae (10.6.9) and (10.6.10) imply that

1. for any function  $f$ ,

$$[f, \partial_{i_1} \wedge \dots \wedge \partial_{i_p}] = \sum_{k=1}^p (-1)^{k-1} (\partial_{i_k} f) \partial_{i_1} \wedge \dots \wedge \check{\partial}_{i_k} \wedge \dots \wedge \partial_{i_p}$$

where  $\check{\phantom{x}}$  over a symbol means that it is deleted, and

2.  $[\partial_{i_1} \wedge \dots \wedge \partial_{i_p}, \partial_{j_1} \wedge \dots \wedge \partial_{j_q}] = 0$ .

Therefore, if

$$A = A^{i_1 \dots i_p} \partial_{i_1} \wedge \dots \wedge \partial_{i_p} \quad \text{and} \quad B = B^{j_1 \dots j_q} \partial_{j_1} \wedge \dots \wedge \partial_{j_q},$$

we get

$$\begin{aligned} [A, B] &= A^{\ell i_1 \dots i_{\ell-1} i_{\ell+1} \dots i_p} \partial_{i_1} \wedge \dots \wedge \partial_{i_{\ell-1}} \wedge \partial_{i_{\ell+1}} \\ &\quad \wedge \partial_{j_1} \wedge \dots \wedge \partial_{j_q} + (-1)^p B^{\ell j_1 \dots j_{\ell-1} j_{\ell+1} \dots j_q} \partial_{i_1} \wedge \dots \wedge \partial_{i_{\ell-1}} \\ &\quad \wedge \partial_{i_{\ell+1}} \wedge \partial_{j_1} \wedge \dots \wedge \partial_{j_{\ell-1}} \wedge \partial_{j_{\ell+1}} \wedge \dots \wedge \partial_{j_q} \end{aligned} \quad (10.6.12)$$

or, more succinctly,

$$\begin{aligned} [A, B]^{k_2 \dots k_{p+q}} &= \varepsilon_{i_2 \dots i_p j_1 \dots j_q}^{k_2 \dots k_{p+q}} A^{\ell i_2 \dots i_p} \frac{\partial}{\partial x^\ell} B^{j_1 \dots j_q} \\ &\quad + (-1)^p \varepsilon_{i_1 \dots i_p j_2 \dots j_q}^{k_2 \dots k_{p+q}} B^{\ell j_2 \dots j_p} \frac{\partial}{\partial x^\ell} A^{i_1 \dots i_q} \end{aligned} \quad (10.6.13)$$



where all components are nonstrict. Here

$$\varepsilon_{j_1 \dots j_{p+q}}^{i_1 \dots i_{p+q}}$$

is the **Kronecker symbol**: it is zero if  $(i_1, \dots, i_{p+q}) \neq (j_1, \dots, j_{p+q})$ , and is 1 (resp.,  $-1$ ) if  $j_1, \dots, j_{p+q}$  is an even (resp., odd) permutation of  $i_1, \dots, i_{p+q}$ .

From §10.6 the Poisson tensor  $B \in \Omega_2(P)$  defined by a Poisson bracket  $\{, \}$  on  $P$  satisfies  $B(\mathbf{d}F, \mathbf{d}G) = \{F, G\}$  for any  $F, G \in \mathcal{F}(P)$ . By (10.6.2), this can be written

$$\{F, G\} = \mathbf{i}_B(\mathbf{d}F \wedge \mathbf{d}G), \quad (10.6.14)$$

or in local coordinates,

$$\{F, G\} = B^{IJ} \frac{\partial F}{\partial z^I} \frac{\partial G}{\partial z^J}.$$

Writing  $B$  locally as a sum of terms of the form  $X \wedge Y$  for some  $X, Y \in \mathfrak{X}(P)$  and taking  $Z \in \mathfrak{X}(P)$  arbitrary, by (10.6.1), we have for  $F, G, H \in \mathcal{F}(P)$ ,

$$\begin{aligned} \mathbf{i}_B(\mathbf{d}F \wedge \mathbf{d}G \wedge \mathbf{d}H)(Z) &= (\mathbf{d}F \wedge \mathbf{d}G \wedge \mathbf{d}H)(X, Y, Z) \\ &= \det \begin{bmatrix} \mathbf{d}F(X) & \mathbf{d}F(Y) & \mathbf{d}F(Z) \\ \mathbf{d}G(X) & \mathbf{d}G(Y) & \mathbf{d}G(Z) \\ \mathbf{d}H(X) & \mathbf{d}H(Y) & \mathbf{d}H(Z) \end{bmatrix} \\ &= \det \begin{bmatrix} \mathbf{d}F(X) & \mathbf{d}F(Y) \\ \mathbf{d}G(X) & \mathbf{d}G(Y) \end{bmatrix} \mathbf{d}H(Z) + \det \begin{bmatrix} \mathbf{d}H(X) & \mathbf{d}H(Y) \\ \mathbf{d}F(X) & \mathbf{d}F(Y) \end{bmatrix} \mathbf{d}G(Z) \\ &\quad + \det \begin{bmatrix} \mathbf{d}G(X) & \mathbf{d}G(Y) \\ \mathbf{d}H(X) & \mathbf{d}H(Y) \end{bmatrix} \mathbf{d}F(Z) \\ &= \mathbf{i}_B(\mathbf{d}F \wedge \mathbf{d}G) \mathbf{d}H(Z) + \mathbf{i}_B(\mathbf{d}H \wedge \mathbf{d}F) \mathbf{d}G(Z) + \mathbf{i}_B(\mathbf{d}G \wedge \mathbf{d}H) \mathbf{d}F(Z), \end{aligned}$$

that is,

$$\begin{aligned} \mathbf{i}_B(\mathbf{d}F \wedge \mathbf{d}G \wedge \mathbf{d}H) &= \mathbf{i}_B(\mathbf{d}F \wedge \mathbf{d}G) \mathbf{d}H + \mathbf{i}_B(\mathbf{d}H \wedge \mathbf{d}F) \mathbf{d}G + \mathbf{i}_B(\mathbf{d}G \wedge \mathbf{d}H) \mathbf{d}F \end{aligned} \quad (10.6.15)$$

**The Jacobi–Schouten Identity.** Equations (10.6.14) and (10.6.15) imply

$$\begin{aligned} &\{\{F, G\}, H\} + \{\{H, F\}, G\} + \{\{G, H\}, F\} \\ &= \mathbf{i}_B(\mathbf{d}\{F, G\} \wedge \mathbf{d}H) + \mathbf{i}_B(\mathbf{d}\{H, F\} \wedge \mathbf{d}G) + \mathbf{i}_B(\mathbf{d}\{G, H\} \wedge \mathbf{d}F) \\ &= \mathbf{i}_B \mathbf{d}(\mathbf{i}_B(\mathbf{d}F \wedge \mathbf{d}G) \mathbf{d}H + \mathbf{i}_B(\mathbf{d}H \wedge \mathbf{d}F) \mathbf{d}G + \mathbf{i}_B(\mathbf{d}G \wedge \mathbf{d}H) \mathbf{d}F) \\ &= \mathbf{i}_B \mathbf{d} \mathbf{i}_B(\mathbf{d}F \wedge \mathbf{d}G \wedge \mathbf{d}H) \\ &= \tfrac{1}{2} \mathbf{i}_{[B, B]}(\mathbf{d}F \wedge \mathbf{d}G \wedge \mathbf{d}H), \end{aligned}$$

the last equality being a consequence of (10.6.11). We summarize what we have proved.

**Theorem 10.6.2.** *The following identity holds.*

$$\begin{aligned} \{\{F, G\}, H\} + \{\{H, F\}, G\} + \{\{G, H\}, F\} \\ = \frac{1}{2} \mathbf{i}_{[B, B]}(\mathbf{d}F \wedge \mathbf{d}G \wedge \mathbf{d}H) \end{aligned} \quad (10.6.16)$$

This result shows that Jacobi's identity for  $\{, \}$  is equivalent to  $[B, B] = 0$ . Thus, a Poisson structure is uniquely defined by a contravariant antisymmetric two-tensor whose Schouten bracket with itself vanishes. The local formula (10.6.13) becomes

$$[B, B]^{IJK} = \sum_{L=1}^n \left( B^{LK} \frac{\partial B^{IJ}}{\partial z^L} + B^{LI} \frac{\partial B^{JK}}{\partial z^L} + B^{LJ} \frac{\partial B^{KI}}{\partial z^L} \right)$$

which coincides with our earlier expression (10.4.2).

**The Lie–Schouten Identity.** There is another interesting identity that gives the Lie derivative of the Poisson tensor along a Hamiltonian vector field.

**Theorem 10.6.3.** *The following identity holds*

$$\mathcal{L}_{X_H} B = \mathbf{i}_{[B, B]} \mathbf{d}H. \quad (10.6.17)$$

**Proof.** In coordinates,

$$(\mathcal{L}_X B)^{IJ} = X^K \frac{\partial B^{IJ}}{\partial z^K} - B^{IK} \frac{\partial X^J}{\partial z^K} - B^{KJ} \frac{\partial X^I}{\partial z^K}$$

so if  $X^I = B^{IJ}(\partial H / \partial z^J)$ , this becomes

$$\begin{aligned} (\mathcal{L}_{X_H} B)^{IJ} &= B^{KL} \frac{\partial B^{IJ}}{\partial z^K} \frac{\partial H}{\partial z^L} - B^{IK} \frac{\partial}{\partial z^K} \left( B^{JL} \frac{\partial H}{\partial z^L} \right) \\ &\quad + B^{JK} \frac{\partial}{\partial z^K} \left( B^{IL} \frac{\partial H}{\partial z^L} \right) \\ &= \left( B^{KL} \frac{\partial B^{IJ}}{\partial z^K} - B^{IK} \frac{\partial B^{JL}}{\partial z^K} - B^{KJ} \frac{\partial B^{IL}}{\partial z^K} \right) \frac{\partial H}{\partial z^L} \\ &= [B, B]^{LIJ} \frac{\partial H}{\partial z^L} = (\mathbf{i}_{[B, B]} \mathbf{d}H)^{IJ}, \end{aligned}$$

so (10.6.17) follows. ■

This identity shows how Jacobi's identity  $[B, B] = 0$  is directly used to show that the flow  $\varphi_t$  of a Hamiltonian vector field is Poisson. The above

derivation shows that *the flow of a time-dependent Hamiltonian vector field consists of Poisson maps*; indeed, even in this case,

$$\frac{d}{dt}(\varphi_t^* B) = \varphi_t^*(\mathcal{L}_{X_H} B) = \varphi_t^*(\mathbf{i}_{[B, B]} \mathbf{d}H) = 0$$

is valid.

### Exercises

◇ **Exercise 10.6-1.** Prove the following formulae by the method indicated in the text.

(a) If  $A \in \Omega_q(P)$  and  $X \in \mathfrak{X}(P)$ , then  $[X, A] = \mathcal{L}_X A$ .

(b) If  $A \in \Omega_q(P)$  and  $X_1, \dots, X_r \in \mathfrak{X}(P)$ , then

$$[X_1 \wedge \dots \wedge X_r, A] = \sum_{i=1}^r (-1)^{i+1} X_1 \wedge \dots \wedge \check{X}_i \wedge \dots \wedge X_r \wedge (\mathcal{L}_{X_i} A).$$

(c) If  $X_1, \dots, X_r, Y_1, \dots, Y_s \in \mathfrak{X}(P)$ , then

$$\begin{aligned} [X_1 \wedge \dots \wedge X_r, Y_1 \wedge \dots \wedge Y_s] \\ = (-1)^{r+1} \sum_{i=1}^r \sum_{j=1}^s (-1)^{i+j} [X_i, Y_j] \wedge \\ \wedge X_1 \wedge \dots \wedge \check{X}_i \wedge \dots \wedge X_r \wedge Y_1 \wedge \dots \wedge \check{Y}_j \wedge \dots \wedge Y_s. \end{aligned}$$

(d) If  $A \in \Omega_p(P)$ ,  $B \in \Omega_q(P)$ , and  $\alpha \in \Omega^{p+q-1}(P)$ , then

$$\mathbf{i}_{[A, B]} \alpha = (-1)^{q(p+1)} \mathbf{i}_A \mathbf{d} \mathbf{i}_B \alpha + (-1)^p \mathbf{i}_B \mathbf{d} \mathbf{i}_A \alpha - \mathbf{i}_B \mathbf{i}_A \mathbf{d} \alpha.$$

◇ **Exercise 10.6-2.** Let  $M$  be a finite-dimensional manifold. A ***k-vector field*** skew-symmetric contravariant tensor field  $A(x) : T_x^* M \times \dots \times T_x^* M \rightarrow \mathbb{R}$  ( $k$  copies of  $T_x^* M$ ). Let  $x_0 \in M$  be such that  $A(x_0) = 0$ .

(a) If  $X \in \mathfrak{X}(M)$ , show that  $(\mathcal{L}_X A)(x_0)$  depends only on  $X(x_0)$ , thereby defining a map  $\mathbf{d}_{x_0} A : T_{x_0} M \rightarrow T_{x_0} M \wedge \dots \wedge T_{x_0} M$  ( $k$  times), called the ***intrinsic derivative*** of  $A$  at  $x_0$ .

(b) If  $\alpha_1, \dots, \alpha_k \in T_x^* M$ ,  $v_1, \dots, v_k \in T_x M$ , show that

$$\langle \alpha_1 \wedge \dots \wedge \alpha_k, v_1 \wedge \dots \wedge v_k \rangle := \det [\langle \alpha_i, v_j \rangle]$$

defines a nondegenerate pairing between  $T_x^* M \wedge \dots \wedge T_x^* M$  and  $T_x M \wedge \dots \wedge T_x M$ . Conclude that these two spaces are dual to each other, that

the space  $\Omega^k(M)$  of  $k$ -forms is dual to the space of  $k$ -contravariant skew-symmetric tensor fields  $\Omega_k(M)$ , and that the bases

$$\{ \mathbf{d}x^{i_1} \wedge \cdots \wedge \mathbf{d}x^{i_k} \mid i_1 < \cdots < i_k \}$$

and

$$\left\{ \frac{\partial}{\partial x^{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{i_k}} \mid i_1 < \cdots < i_k \right\}$$

are dual to each other.

(c) Show that the dual map

$$(\mathbf{d}_{x_0}A)^* : T_{x_0}^*M \wedge \cdots \wedge T_{x_0}^*M \rightarrow T_{x_0}^*M, A(x_0) = 0,$$

is given by

$$(\mathbf{d}_{x_0}A)^*(\alpha_1 \wedge \cdots \wedge \alpha_k) = \mathbf{d}(A(\tilde{\alpha}_1, \dots, \tilde{\alpha}_k))(x_0),$$

where  $\tilde{\alpha}_1, \dots, \tilde{\alpha}_k \in \Omega^1(M)$  are arbitrary one-forms whose values at  $x_0$  are  $\alpha_1, \dots, \alpha_k$ .

◇ **Exercise 10.6-3 (Weinstein [1983]).** Let  $(P, \{ , \})$  be a finite-dimensional Poisson manifold with Poisson tensor  $B \in \Omega_2(P)$ . Let  $z_0 \in P$  be such that  $B(z_0) = 0$ . For  $\alpha, \beta \in T_{z_0}^*P$ , define

$$[\alpha, \beta]_B = (\mathbf{d}_{z_0}B)^*(\alpha \wedge \beta) = \mathbf{d}(B(\tilde{\alpha}, \tilde{\beta}))(z_0)$$

where  $\mathbf{d}_{z_0}B$  is the intrinsic derivative of  $B$  and  $\tilde{\alpha}, \tilde{\beta} \in \Omega^1(P)$  are such that  $\tilde{\alpha}(z_0) = \alpha, \tilde{\beta}(z_0) = \beta$ . (See Exercise 10.8-2.) Show that  $(\alpha, \beta) \mapsto [\alpha, \beta]_B$  defines a bilinear skew-symmetric map  $T_{z_0}^*P \times T_{z_0}^*P \rightarrow T_{z_0}^*P$ . Show that the Jacobi identity for the Poisson bracket implies that  $[ , ]_B$  is a Lie bracket on  $T_{z_0}^*P$ . Since  $(T_{z_0}^*P, [ , ]_B)$  is a Lie algebra, its dual  $T_{z_0}P$  naturally carries the induced Lie–Poisson structure, called the *linearization* of the given Poisson bracket at  $z_0$ . Show that the linearization in local coordinates has the expression

$$\{F, G\}(v) = \frac{\partial B^{ij}(z_0)}{\partial z^k} \frac{\partial F}{\partial v^i} \frac{\partial G}{\partial v^j} v^k,$$

for  $F, G : T_{z_0}P \rightarrow \mathbb{R}$  and  $v \in T_{z_0}P$ .

◇ **Exercise 10.6-4 (Magri–Weinstein).** On the finite-dimensional manifold  $P$ , assume one has a symplectic form  $\Omega$  and a Poisson structure  $B$ . Denote by  $K = B^\sharp \circ \Omega^\flat : TP \rightarrow TP$ . Show that  $(\Omega^\flat)^{-1} + B^\sharp : T^*P \rightarrow TP$  defines a new Poisson structure on  $P$  if and only if  $\Omega^\flat \circ K^n$  induces a presymplectic form on  $P$  for all  $n \in \mathbb{N}$ .

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## 10.7 Generalities on Lie–Poisson Structures

**The Lie–Poisson Equations.** We begin by working out Hamilton’s equations for the Lie–Poisson bracket.

**Proposition 10.7.1.** *Let  $G$  be a Lie group. The equations of motion for the Hamiltonian  $H$  with respect to the  $\pm$  Lie–Poisson brackets on  $\mathfrak{g}^*$  are*

$$\frac{d\mu}{dt} = \mp \operatorname{ad}_{\delta H/\delta \mu}^* \mu. \quad (10.7.1)$$

**Proof.** Let  $F \in \mathcal{F}(\mathfrak{g}^*)$  be an arbitrary function. By the chain rule,

$$\frac{dF}{dt} = \mathbf{D}F(\mu) \cdot \dot{\mu} = \left\langle \dot{\mu}, \frac{\delta F}{\delta \mu} \right\rangle, \quad (10.7.2)$$

while

$$\begin{aligned} \{F, H\}_{\pm}(\mu) &= \pm \left\langle \mu, \left[ \frac{\delta F}{\delta \mu}, \frac{\delta H}{\delta \mu} \right] \right\rangle = \pm \left\langle \mu, -\operatorname{ad}_{\delta H/\delta \mu} \frac{\delta F}{\delta \mu} \right\rangle \\ &= \mp \left\langle \operatorname{ad}_{\delta H/\delta \mu}^* \mu, \frac{\delta F}{\delta \mu} \right\rangle. \end{aligned} \quad (10.7.3)$$

Nondegeneracy of the pairing and arbitrariness of  $F$  imply the result. ■

**Caution.** In infinite dimensions,  $\mathfrak{g}^*$  does not necessarily mean the literal functional analytic dual of  $\mathfrak{g}$ , but rather a space in (nondegenerate) duality with  $\mathfrak{g}$ . In this case, care must be taken with the definition of  $\delta F/\delta \mu$ . ♦

Formula (10.7.1) says that on  $\mathfrak{g}_{\pm}^*$ , the Hamiltonian vector field of  $H : \mathfrak{g}^* \rightarrow \mathbb{R}$  is given by

$$X_H(\mu) = \mp \operatorname{ad}_{\delta H/\delta \mu}^* \mu. \quad (10.7.4)$$

For example, for  $G = \operatorname{SO}(3)$ , formula (10.1.3) for the Lie–Poisson bracket gives

$$X_H(\Pi) = \Pi \times \nabla H. \quad (10.7.5)$$

**Historical Note.** Lagrange devoted a good deal of attention in Volume 2 of *Mécanique Analytique* [1788] to the study of rotational motion of mechanical systems. In fact, in equation A on page 212 he gives the reduced Lie–Poisson equations for  $\operatorname{SO}(3)$  for a rather general Lagrangian. This equation is essentially the same as (10.7.5). His derivation was just how we would do it today—by reduction from material to spatial representation. Formula (10.7.5) actually hides a subtle point in that it identifies  $\mathfrak{g}$  and  $\mathfrak{g}^*$ . Indeed, the way Lagrange wrote the equations, they are much more like their counterpart on  $\mathfrak{g}$ , which are called the *Euler–Poincaré equations*. We will come to these in Chapter 13, where additional historical information may be found.

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**Coordinate Formulas.** In finite dimensions, if  $\xi_a$ ,  $a = 1, 2, \dots, l$ , is a basis for  $\mathfrak{g}$ , the structure constants  $C_{ab}^d$  are defined by

$$[\xi_a, \xi_b] = C_{ab}^d \xi_d \quad (10.7.6)$$

(a sum on “ $d$ ” is understood). Thus, the Lie–Poisson bracket becomes

$$\{F, K\}_{\pm}(\mu) = \pm \mu_d \frac{\partial F}{\partial \mu_a} \frac{\partial K}{\partial \mu_b} C_{ab}^d, \quad (10.7.7)$$

where  $\mu = \mu_a \xi^a$ ,  $\{\xi^a\}$  is the basis of  $\mathfrak{g}^*$  dual to  $\{\xi_a\}$ , and summation on repeated indices is understood. Taking  $F$  and  $K$  to be components of  $\mu$ , (10.7.7) becomes

$$\{\mu_a, \mu_b\}_{\pm} = \pm C_{ab}^d \mu_d. \quad (10.7.8)$$

The equations of motion for a Hamiltonian  $H$  likewise become

$$\dot{\mu}_a = \mp \mu_d C_{ab}^d \frac{\partial H}{\partial \mu_b}. \quad (10.7.9)$$

**Poisson Maps.** In the Lie–Poisson reduction theorem in Chapter 13 we will show that the maps from  $T^*G$  to  $\mathfrak{g}^*$  (resp.,  $\mathfrak{g}_+^*$ ) given by  $\alpha_g \mapsto T_e^* L_g \cdot \alpha_g$  (resp.,  $\alpha_g \mapsto T_e^* R_g \cdot \alpha_g$ ) are Poisson maps. We will show in the next chapter that this is a general property of momentum maps. Here is another class of Poisson maps that will also turn out to be momentum maps.

**Proposition 10.7.2.** *Let  $G$  and  $H$  be Lie groups and let  $\mathfrak{g}$  and  $\mathfrak{h}$  be their Lie algebras. Let  $\alpha : \mathfrak{g} \rightarrow \mathfrak{h}$  be a linear map. The map  $\alpha$  is a homomorphism of Lie algebras if and only if its dual  $\alpha^* : \mathfrak{h}_{\pm}^* \rightarrow \mathfrak{g}_{\pm}^*$  is a (linear) Poisson map.*

**Proof.** Let  $F, K \in \mathcal{F}(\mathfrak{g}^*)$ . To compute  $\delta(F \circ \alpha^*)/\delta\mu$ , we let  $\nu = \alpha^*(\mu)$  and use the definition of the functional derivative and the chain rule to get

$$\begin{aligned} \left\langle \frac{\delta}{\delta\mu}(F \circ \alpha^*), \delta\mu \right\rangle &= \mathbf{D}(F \circ \alpha^*)(\mu) \cdot \delta\mu = \mathbf{D}F(\alpha^*(\mu)) \cdot \alpha^*(\delta\mu) \\ &= \left\langle \alpha^*(\delta\mu), \frac{\delta F}{\delta\nu} \right\rangle = \left\langle \delta\mu, \alpha \cdot \frac{\delta F}{\delta\nu} \right\rangle. \end{aligned} \quad (10.7.10)$$

Thus,

$$\frac{\delta}{\delta\mu}(F \circ \alpha^*) = \alpha \cdot \frac{\delta F}{\delta\nu}. \quad (10.7.11)$$

Hence,

$$\begin{aligned} \{F \circ \alpha^*, K \circ \alpha^*\}_{\pm}(\mu) &= \left\langle \mu, \left[ \frac{\delta}{\delta\mu}(F \circ \alpha^*), \frac{\delta}{\delta\mu}(K \circ \alpha^*) \right] \right\rangle \\ &= \left\langle \mu, \left[ \alpha \cdot \frac{\delta F}{\delta\nu}, \alpha \cdot \frac{\delta K}{\delta\nu} \right] \right\rangle. \end{aligned} \quad (10.7.12)$$

The expression (10.7.12) equals

$$\left\langle \mu, \alpha \cdot \left[ \frac{\delta F}{\delta \nu}, \frac{\delta G}{\delta \nu} \right] \right\rangle \quad (10.7.13)$$

for all  $F$  and  $K$  if and only if  $\alpha$  is a Lie algebra homomorphism. ■

This theorem applies to the case  $\alpha = T_e \sigma$  for  $\sigma : G \rightarrow H$  a Lie group homomorphism, by studying the reduction diagram in Figure 10.9.1 (and being cautious that  $\sigma$  need not be a diffeomorphism.)

$$\begin{array}{ccc} T_g^* G & \xleftarrow{T^* \sigma} & T_{\sigma(g)}^* H \\ \text{right} \downarrow & & \downarrow \text{right} \\ \text{translate to} & & \text{translate to} \\ \text{identity} & & \text{identity} \\ \mathfrak{g}_+^* & \xleftarrow{\alpha^*} & \mathfrak{h}_+^* \end{array}$$

FIGURE 10.7.1. Lie group homomorphisms induce Poisson maps.

## Examples

### (a) Plasma to Fluid Poisson Map for the Momentum Variables.

Let  $G$  be the group of diffeomorphisms of a manifold  $Q$  and let  $H$  be the group of canonical transformations of  $P = T^*Q$ . We assume that  $H^1(Q) = 0$  which implies that all locally Hamiltonian vector fields on  $T^*Q$  are globally Hamiltonian. Thus, the Lie algebra  $\mathfrak{h}$  consists of functions on  $T^*Q$  modulo constants. Its dual is identified with itself via the  $L^2$ -inner product relative to the Liouville measure  $dq dp$  on  $T^*Q$ . Let  $\sigma : G \rightarrow H$  be the map  $\eta \mapsto T^* \eta^{-1}$ , which is a group homomorphism and let  $\alpha = T_e \sigma : \mathfrak{g} \rightarrow \mathfrak{h}$ . We claim that  $\alpha^* : \mathcal{F}(T^*Q)/\mathbb{R} \rightarrow \mathfrak{g}^*$  is given by

$$\alpha^*(F) = \int p f(q, p) dp, \quad (10.7.14)$$

where we regard  $\mathfrak{g}^*$  as the space of one-form densities on  $Q$  and the integral denotes fiber integration for each fixed  $q \in Q$ . Indeed,  $\alpha$  is the map taking vector fields  $X$  on  $Q$  to their lifts  $X_{\mathcal{P}(X)}$  on  $T^*Q$ . Thus, as a map of  $\mathfrak{X}(Q)$  to  $\mathcal{F}(T^*Q)/\mathbb{R}$ ,  $\alpha$  is given by  $X \mapsto \mathcal{P}(X)$ . Its dual is given by

$$\begin{aligned} \langle \alpha^*(f), X \rangle &= \langle f, \alpha(X) \rangle = \int_P f \mathcal{P}(X) dq dp \\ &= \int_P f(q, p) p \cdot X(q) dq dp \end{aligned} \quad (10.7.15)$$

so  $\alpha^*(F)$  is given by (10.7.14) as claimed.  $\blacklozenge$

**(b) Plasma to Fluid Map for the Density Variable.** Let  $G = \mathcal{F}(Q)$ , regarded as an abelian group and let the map  $\sigma : G \rightarrow \text{Diff}_{\text{can}}(T^*Q)$  be given by  $\sigma(\varphi) = \text{fiber translation by } \mathbf{d}\varphi$ . A computation similar to that above gives the Poisson map

$$\alpha^*(f)(q) = \int f(q, p) dp \quad (10.7.16)$$

from  $\mathcal{F}(T^*Q)$  to  $\text{Den}(Q) = \mathcal{F}(Q)^*$ . The integral in (10.7.16) denotes the fiber integration of  $f(q, p)$  for fixed  $q \in Q$ .  $\blacklozenge$

**Linear Poisson Structures Are Lie–Poisson.** Next we characterize Lie–Poisson brackets as the linear ones. Let  $V^*$  and  $V$  be Banach spaces and let  $\langle \cdot, \cdot \rangle : V^* \times V \rightarrow \mathbb{R}$  be a weakly nondegenerate pairing of  $V^*$  with  $V$ . Think of elements of  $V$  as linear functionals on  $V^*$ . A Poisson bracket on  $V^*$  is called **linear** if the bracket of any two linear functionals on  $V^*$  is again linear. This condition is equivalent to the associated Poisson tensor  $B(\mu) : V \rightarrow V^*$  being *linear* in  $\mu \in V^*$ .

**Proposition 10.7.3.** *Let  $\langle \cdot, \cdot \rangle : V^* \times V \rightarrow \mathbb{R}$  be a (weakly) nondegenerate pairing of the Banach spaces  $V^*$  and  $V$ , and let  $V^*$  have a linear Poisson bracket. Assume that the bracket of any two linear functionals on  $V^*$  is in the range of  $\langle \mu, \cdot \rangle$  for all  $\mu \in V^*$  (this condition is automatically satisfied if  $V$  is finite dimensional). Then  $V$  is a Lie algebra and the Poisson bracket on  $V^*$  is the corresponding Lie–Poisson bracket.*

**Proof.** If  $x \in V$ , we denote by  $x'$  the functional  $x'(\mu) = \langle \mu, x \rangle$  on  $V^*$ . By hypothesis, the Poisson bracket  $\{x', y'\}$  is a linear functional on  $V^*$ . By assumption this bracket is represented by an element which we denote  $[x, y]'$  in  $V$ , that is, we can write  $\{x', y'\} = [x, y]'$ . (The element  $[x, y]$  is unique since  $\langle \cdot, \cdot \rangle$  is weakly nondegenerate.) It is straightforward to check that the operation  $[\cdot, \cdot]$  on  $V$  so defined is a Lie algebra bracket. Thus,  $V$  is a Lie algebra, and one then checks that the given Poisson bracket is the Lie–Poisson bracket for this algebra.  $\blacksquare$

## Exercises

- ◇ **Exercise 10.7-1.** Let  $\sigma : \text{SO}(3) \rightarrow \text{GL}(3)$  be the inclusion map. Identify  $\mathfrak{so}(3)^* = \mathbb{R}^3$  with the rigid body bracket and identify  $\mathfrak{gl}(3)^*$  with  $\mathfrak{gl}(3)$  using  $\langle A, B \rangle = \text{trace}(AB^t)$ . Compute the induced map  $\alpha^* : \mathfrak{gl}(3) \rightarrow \mathbb{R}^3$  and verify directly that it is Poisson.