15
The Free Rigid Body

As an application of the theory developed so far, we discuss the motion of a free rigid body about a fixed point. We begin with a discussion of the kinematics of rigid body motion. Our description of the kinematics of rigid bodies follows some of the notations and conventions of continuum mechanics, as in Marsden and Hughes [1983].

15.1 Material, Spatial, and Body Coordinates

Consider a rigid body, free to move in $\mathbb{R}^3$. A ***reference configuration*** $\mathcal{B}$ of the body is the closure of an open set in $\mathbb{R}^3$ with a piecewise smooth boundary. Points in $\mathcal{B}$, denoted $X = (X^1, X^2, X^3) \in \mathcal{B}$ relative to an orthonormal basis $(\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3)$ are called ***material points*** and $X^i, i = 1, 2, 3$, are called ***material coordinates***. A ***configuration*** of $\mathcal{B}$ is a mapping $\varphi : \mathcal{B} \to \mathbb{R}^3$ which is, for our purposes, C^1, orientation preserving, and invertible on its image. Points in the image of φ are called ***spatial points*** and denoted by lowercase letters. Let $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ be a right-handed orthonormal basis of $\mathbb{R}^3$. Coordinates for spatial points, such as $x = (x^1, x^2, x^3) \in \mathbb{R}^3, i = 1, 2, 3$, relative to the basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ are called ***spatial coordinates.*** Dually, one can consider material quantities such as maps defined on $\mathcal{B}$, say $Z : \mathcal{B} \to \mathbb{R}$. Then we can form spatial quantities by composition: $z_t = Z_t \circ \varphi_t^{-1}$. Spatial quantities are also called ***Eulerian quantities*** and material quantities are often called ***Lagrangian quantities.***

A ***motion*** of $\mathcal{B}$ is a time-dependent family of configurations, written $x = \varphi(X,t) = \varphi_t(X)$ or simply $x(X,t)$ or $x_t(X)$. Spatial quantities are functions of x, and are typically written as lowercase letters. By composition with φ_t, spatial quantities become functions of the material points X.

Rigidity of the body means that the distances between points of the body are fixed as the body moves. We shall assume that no external forces act on the body and that the center of mass is fixed at the origin (see Exercise 15.1-1). Since any isometry of $\mathbb{R}^3$ that leaves the origin fixed is a rotation (a 1932 theorem of Mazur and Ulam), we can write

$$x(X,t) = \mathbf{R}(t)X, \quad \text{i.e.,} \quad x^i = \mathbf{R}^i_j(t)X^j, \quad i,j = 1,2,3, \text{ sum on } j,$$

where x^i are the components of x relative to the basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ fixed in space, and $[\mathbf{R}^i_j]$ is the matrix of $\mathbf{R}$ relative to the basis $(\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3)$ and $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$. The motion is assumed to be continuous and $\mathbf{R}(0)$ is the identity, so $\det(\mathbf{R}(t)) = 1$ and thus $\mathbf{R}(t) \in \mathrm{SO}(3)$, the proper orthogonal group. Thus, *the configuration space for the rotational motion of a rigid body may be identified with* $\mathrm{SO}(3)$. *Consequently, the velocity phase space of the free rigid body is* $T\,\mathrm{SO}(3)$ *and the momentum phase space is the cotangent bundle* $T^*\,\mathrm{SO}(3)$. Euler angles*, discussed shortly, are the traditional way to parametrize* $\mathrm{SO}(3)$.

In addition to the material and spatial coordinates, there is a third set, the *convected* or *body coordinates.* These are the coordinates associated with the moving basis, and the description of the rigid body motion in these coordinates, due to Euler, becomes very simple. As before, let $\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3$ be an orthonormal basis fixed in the reference configuration. Let the time-dependent basis $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3$ be defined by $\boldsymbol{\xi}_i = \mathbf{R}(t)\mathbf{E}_i$, $i = 1,2,3$, so $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3$ move attached to the body. The ***body coordinates*** of a vector in $\mathbb{R}^3$ are its components relative to $\boldsymbol{\xi}_i$. For the rigid body anchored at the origin and rotating in space, $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is thought of as a basis fixed in space, whereas $(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3)$ is a basis fixed in the body and moving with it. For this reason $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is called the ***spatial coordinate system*** and $(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3)$ the ***body coordinate system***.

Exercises

◇ **Exercise 15.1-1.** Start with SE(3) as the configuration space for the rigid body and "reduce out" (see §10.7, the Euler–Poincaré, and Lie–Poisson reduction theorems) translations to arrive at SO(3) as the configuration space.

15.2 The Lagrangian of the Free Rigid Body

If $X \in \mathcal{B}$ is a material point of the body, the corresponding trajectory followed by X in space is $x(t) = \mathbf{R}(t)X$, where $\mathbf{R}(t) \in \mathrm{SO}(3)$. The ***material*** or ***Lagrangian velocity*** $V(X,t)$ is defined by

$$V(X,t) = \frac{\partial x(X,t)}{\partial t} = \dot{\mathbf{R}}(t)X, \tag{15.2.1}$$

while the ***spatial*** or ***Eulerian velocity*** $v(x,t)$ is defined by

$$v(x,t) = V(X,t) = \dot{\mathbf{R}}(t)\mathbf{R}(t)^{-1}x, \tag{15.2.2}$$

and the ***body*** or ***convective velocity*** $\mathcal{V}(X,t)$ is defined by taking the velocity regarding X as time-dependent and x fixed, that is, $X(x,t) = \mathbf{R}(t)^{-1}x$:

$$\begin{aligned}\mathcal{V}(X,t) = -\frac{\partial X(x,t)}{\partial t} &= \mathbf{R}(t)^{-1}\dot{\mathbf{R}}(t)\mathbf{R}(t)^{-1}x \\ &= \mathbf{R}(t)^{-1}\dot{\mathbf{R}}(t)X \\ &= \mathbf{R}(t)^{-1}V(X,t) \\ &= \mathbf{R}(t)^{-1}v(x,t). \end{aligned} \tag{15.2.3}$$

Assume that the mass distribution of the body is described by a compactly supported density measure $\rho_0 d^3X$ in the reference configuration, which is zero at points outside the body. The Lagrangian, taken to be the kinetic energy, is given by any of the following expressions that are related to one another by a change of variables and the identities $\|\mathcal{V}\| = \|V\| = \|v\|$:

$$L = \frac{1}{2}\int_{\mathcal{B}} \rho_0(X)\|V(X,t)\|^2\, d^3X \qquad \text{(material)} \tag{15.2.4}$$

$$= \frac{1}{2}\int_{\mathrm{R}(t)\mathcal{B}} \rho_0(\mathbf{R}(t)^{-1}x)\|v(x,t)\|^2\, d^3x \quad \text{(spatial)} \tag{15.2.5}$$

$$= \frac{1}{2}\int_{\mathcal{B}} \rho_0(X)\|\mathcal{V}(X,t)\|^2\, d^3X \qquad \text{(convective or body).} \tag{15.2.6}$$

Differentiating $\mathbf{R}(t)^T\mathbf{R}(t) =$ Identity and $\mathbf{R}(t)\mathbf{R}(t)^T =$ Identity with respect to t, it follows that both $\mathbf{R}(t)^{-1}\dot{\mathbf{R}}(t)$ and $\dot{\mathbf{R}}(t)\mathbf{R}(t)^{-1}$ are skew-symmetric. Moreover, by (15.2.2), (15.2.3), and the classical definition $\mathbf{v} = \boldsymbol{\omega}\times\mathbf{r} = \hat{\boldsymbol{\omega}}\mathbf{r}$ of angular velocity, it follows that the vectors $\boldsymbol{\omega}(t)$ and $\boldsymbol{\Omega}(t)$ in $\mathbb{R}^3$ defined by

$$\hat{\boldsymbol{\omega}}(t) = \dot{\mathbf{R}}(t)\mathbf{R}(t)^{-1} \tag{15.2.7}$$

and

$$\hat{\boldsymbol{\Omega}}(t) = \mathbf{R}(t)^{-1}\dot{\mathbf{R}}(t) \tag{15.2.8}$$

represent the ***spatial*** and ***convective angular velocities*** of the body. Note that $\boldsymbol{\omega}(t) = \mathbf{R}(t)\boldsymbol{\Omega}(t)$, or as matrices,

$$\hat{\boldsymbol{\omega}} = \mathrm{Ad}_{\mathrm{R}}\,\hat{\boldsymbol{\Omega}} = \mathbf{R}\hat{\boldsymbol{\Omega}}\mathbf{R}^{-1}.$$

Let us show that $L : T\,\mathrm{SO}(3) \to \mathbb{R}$ given by (15.2.4) is left-invariant. Indeed, if $\mathbf{B} \in \mathrm{SO}(3)$, left translation by $\mathbf{B}$ is $L_{\mathrm{B}}\mathbf{R} = \mathbf{B}\mathbf{R}$ and $TL_{\mathrm{B}}(\mathbf{R}, \dot{\mathbf{R}}) = (\mathbf{B}\mathbf{R}, \mathbf{B}\dot{\mathbf{R}})$, so

$$\begin{aligned} L(TL_{\mathrm{B}}(\mathbf{R}, \dot{\mathbf{R}})) &= \frac{1}{2}\int_{\mathcal{B}} \rho_0(X) \|\mathbf{B}\dot{\mathbf{R}}(X)\|^2 \, d^3X \\ &= \frac{1}{2}\int_{\mathcal{B}} \rho_0(X) \|\dot{\mathbf{R}}(X)\|^2 \, d^3X = L(\mathbf{R}, \dot{\mathbf{R}}) \end{aligned} \tag{15.2.9}$$

since $\mathbf{R}$ is orthogonal.

By Lie–Poisson reduction of dynamics (Chapter 13), the corresponding Hamiltonian system on $T^*\,\mathrm{SO}(3)$, which is necessarily also left invariant, induces a Lie–Poisson system on $\mathfrak{so}(3)^*$ and this system leaves invariant the coadjoint orbits $\|\Pi\| =$ constant. Alternatively, by Euler–Poincaré reduction of dynamics, we get a system of equations in terms of body angular velocity on $so(3)$.

Reconstruction of the dynamics on $T\,\mathrm{SO}(3)$ is simply this: given $\hat{\mathbf{\Omega}}(t)$, determine $\mathbf{R}(t) \in \mathrm{SO}(3)$ from (15.2.8):

$$\dot{\mathbf{R}}(t) = \mathbf{R}(t)\hat{\mathbf{\Omega}}(t), \tag{15.2.10}$$

which is a time-dependent linear equation for $\mathbf{R}(t)$.

15.3 The Lagrangian and Hamiltonian for the Rigid Body in Body Representation

From (15.2.6), (15.2.3), and (15.2.8) of the previous section, the rigid body Lagrangian is

$$L = \frac{1}{2}\int_{\mathcal{B}} \rho_0(X) \|\mathbf{\Omega} \times X\|^2 \, d^3X. \tag{15.3.1}$$

Introducing the new inner product

$$\langle\langle \mathbf{a}, \mathbf{b} \rangle\rangle := \int_{\mathcal{B}} \rho_0(X)(\mathbf{a} \times X) \cdot (\mathbf{b} \times X) \, d^3X,$$

which is determined by the density $\rho_0(X)$ of the body, (15.3.1) becomes

$$L(\mathbf{\Omega}) = \frac{1}{2}\langle\langle \mathbf{\Omega}, \mathbf{\Omega} \rangle\rangle. \tag{15.3.2}$$

Define the linear isomorphism $\mathbf{l} : \mathbb{R}^3 \to \mathbb{R}^3$ by $\mathbf{l}\mathbf{a} \cdot \mathbf{b} = \langle\langle \mathbf{a}, \mathbf{b} \rangle\rangle$ for all $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$; this is possible and uniquely determines $\mathbf{l}$, since both the dot product and $\langle\!\langle\, , \rangle\!\rangle$ are nondegenerate bilinear forms (assuming the rigid body is not concentrated on a line). It is clear that $\mathbf{l}$ is symmetric with respect to the dot product and is positive-definite. Let $(\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3)$ be an orthonormal basis for material coordinates. The matrix of $\mathbf{l}$ is

$$\mathbf{l}_{ij} = \mathbf{E}_i \cdot \mathbf{l}\mathbf{E}_j = \langle\!\langle \mathbf{E}_i, \mathbf{E}_j \rangle\!\rangle = \begin{cases} -\displaystyle\int_{\mathcal{B}} \rho_0(X) X^i X^j \, d^3X, & i \neq j, \\ \displaystyle\int_{\mathcal{B}} \rho_0(X)(\|X\|^2 - (X^i)^2) \, d^3X, & i = j, \end{cases}$$

which are the classical expressions of the matrix of the ***inertia tensor***. Since $\mathbf{l}$ is symmetric, it can be diagonalized; an orthonormal basis in which it is diagonal is a ***principal axis body frame*** and the diagonal elements I_1, I_2, I_3 are the ***principal moments of inertia*** of the rigid body. In what follows we work in a principal axis reference and body frame, $(\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3)$.

Since $\mathfrak{so}(3)^*$ and $\mathbb{R}^3$ are identified by the dot product (not by $\langle\!\langle\, , \rangle\!\rangle$), the linear functional $\langle\!\langle \mathbf{\Omega}, \cdot \rangle\!\rangle$—the Legendre transformation of $\mathbf{\Omega}$—on $\mathfrak{so}(3) \cong \mathbb{R}^3$ is identified with $\mathbf{l\Omega} := \mathbf{\Pi} \in \mathfrak{so}(3)^* \cong \mathbb{R}^3$ because $\mathbf{\Pi} \cdot \mathbf{a} = \langle\!\langle \mathbf{\Omega}, \mathbf{a} \rangle\!\rangle$ for all $\mathbf{a} \in \mathbb{R}^3$. With $\mathbf{l} = \operatorname{diag}(I_1, I_2, I_3)$, (15.3.2) defines a function

$$K(\mathbf{\Pi}) = \frac{1}{2}\left(\frac{\Pi_1^2}{I_1} + \frac{\Pi_2^2}{I_2} + \frac{\Pi_3^2}{I_3}\right) \tag{15.3.3}$$

that represents the expression for the kinetic energy on $\mathfrak{so}(3)^*$; note that $\mathbf{\Pi} = \mathbf{l\Omega}$ is the ***angular momentum in the body frame***. Indeed, for any $\mathbf{a} \in \mathbb{R}^3$, the identity $(X \times (\mathbf{\Omega} \times X)) \cdot \mathbf{a} = (\mathbf{\Omega} \times X) \cdot (\mathbf{a} \times X)$ and the classical expression of the angular momentum in the body frame, namely,

$$\int_{\mathcal{B}} (X \times \mathcal{V}) \rho_0(X) \, d^3X \tag{15.3.4}$$

gives

$$\begin{aligned} \left(\int_{\mathcal{B}} (X \times \mathcal{V}) \rho_0(X) \, d^3X\right) \cdot \mathbf{a} &= \int_{\mathcal{B}} (X \times (\mathbf{\Omega} \times X)) \cdot \mathbf{a} \rho_0(X) \, d^3X \\ &= \int_{\mathcal{B}} (\mathbf{\Omega} \times X) \cdot (\mathbf{a} \times X) \rho_0(X) \, d^3X \\ &= \langle\!\langle \mathbf{\Omega}, \mathbf{a} \rangle\!\rangle = \mathbf{l\Omega} \cdot \mathbf{a} = \mathbf{\Pi} \cdot \mathbf{a}, \end{aligned}$$

that is, the expression (15.3.4) equals $\mathbf{\Pi}$.

The ***angular momentum in space*** has the expression

$$\boldsymbol{\pi} = \int_{\mathrm{R}(\mathcal{B})} (x \times v) \rho(x) \, d^3x, \tag{15.3.5}$$

where $\rho(x) = \rho_0(X)$ is the ***spatial mass density*** and $v = \boldsymbol{\omega} \times x$ is the spatial velocity (see (15.2.2) and (15.2.7)). For any $\mathbf{a} \in \mathbb{R}^3$,

$$\begin{aligned}\boldsymbol{\pi} \cdot \mathbf{a} &= \int_{\mathrm{R}(\mathcal{B})} (x \times (\boldsymbol{\omega} \times x)) \cdot \mathbf{a}\rho(x)\, d^3X \\ &= \int_{\mathrm{R}(\mathcal{B})} (\boldsymbol{\omega} \times x) \cdot (\mathbf{a} \times x)\rho(x)\, d^3X. \end{aligned} \tag{15.3.6}$$

Changing variables $x = \mathbf{R}X$, (15.3.6) becomes

$$\begin{aligned}&\int_{\mathcal{B}} (\boldsymbol{\omega} \times \mathbf{R}X) \cdot (\mathbf{a} \times \mathbf{R}X)\rho_0(X)\, d^3X \\ &\qquad = \int_{\mathcal{B}} (\mathbf{R}^T\boldsymbol{\omega} \times X) \cdot (\mathbf{R}^T\mathbf{a} \times X)\rho_0(X)\, d^3X \\ &\qquad = \langle\!\langle \boldsymbol{\Omega}, \mathbf{R}^T\mathbf{a} \rangle\!\rangle = \boldsymbol{\Pi} \cdot \mathbf{R}^T\mathbf{a} = \mathbf{R}\boldsymbol{\Pi} \cdot \mathbf{a},\end{aligned}$$

that is,

$$\boldsymbol{\pi} = \mathbf{R}\boldsymbol{\Pi}. \tag{15.3.7}$$

Since L given by (15.3.2) is left invariant on $T\,\mathrm{SO}(3)$, the function K defined on $\mathfrak{so}(3)^*$ by (15.3.3) defines the Lie–Poisson equations of motion on $\mathfrak{so}(3)^*$ relative to the bracket

$$\{F, H\}(\boldsymbol{\Pi}) = -\boldsymbol{\Pi} \cdot (\nabla F(\boldsymbol{\Pi}) \times \nabla H(\boldsymbol{\Pi})). \tag{15.3.8}$$

Since $\nabla K(\boldsymbol{\Pi}) = \mathbf{l}^{-1}\boldsymbol{\Pi}$, we get from (15.3.8) the rigid body equations

$$\dot{\boldsymbol{\Pi}} = -\nabla K(\boldsymbol{\Pi}) \times \boldsymbol{\Pi} = \boldsymbol{\Pi} \times \mathbf{l}^{-1}\boldsymbol{\Pi}, \tag{15.3.9}$$

that is, they are the standard ***Euler equations***:

$$\begin{aligned}\dot{\Pi}_1 &= \frac{I_2 - I_3}{I_2 I_3}\Pi_2\Pi_3, \\ \dot{\Pi}_2 &= \frac{I_3 - I_1}{I_1 I_3}\Pi_1\Pi_3, \\ \dot{\Pi}_3 &= \frac{I_1 - I_2}{I_1 I_2}\Pi_1\Pi_2.\end{aligned} \tag{15.3.10}$$

The fact that these equations preserve coadjoint orbits amounts, in this case, to the easily verified fact that

$$\Pi^2 := \|\boldsymbol{\Pi}\|^2 \tag{15.3.11}$$

is a constant of the motion. In terms of coadjoint orbits, these equations are Hamiltonian on each sphere in $\mathbb{R}^3$ with Hamiltonian function K. The functions

$$C_\Phi(\boldsymbol{\Pi}) = \Phi\left(\frac{1}{2}\|\boldsymbol{\Pi}\|^2\right), \tag{15.3.12}$$

for any $\Phi : \mathbb{R} \to \mathbb{R}$, are easily seen to be Casimir functions.

The conserved momentum resulting from left invariance is the ***spatial angular momentum***:

$$\boldsymbol{\pi} = \mathbf{R}\boldsymbol{\Pi}. \tag{15.3.13}$$

Using left invariance, or a direct calculation, one finds that $\boldsymbol{\pi}$ is constant in time. Indeed,

$$\begin{aligned}\dot{\boldsymbol{\pi}} &= (\mathbf{R}\boldsymbol{\Pi})^{\cdot} = \dot{\mathbf{R}}\boldsymbol{\Pi} + \mathbf{R}\dot{\boldsymbol{\Pi}} = \boldsymbol{\omega} \times \mathbf{R}\boldsymbol{\Pi} + \mathbf{R}\dot{\boldsymbol{\Pi}} \\ &= \mathbf{R}\boldsymbol{\Omega} \times \mathbf{R}\boldsymbol{\Pi} + \mathbf{R}\dot{\boldsymbol{\Pi}} = \mathbf{R}(-\boldsymbol{\Pi} \times \mathbb{I}^{-1}\boldsymbol{\Pi} + \dot{\boldsymbol{\Pi}}) = 0.\end{aligned}$$

The flow lines are given by intersecting the ellipsoids $K =$ constant with the coadjoint orbits which are two-spheres. For distinct moments of inertia $I_1 > I_2 > I_3$, the flow on the sphere has saddle points at $(0, \pm\Pi, 0)$ and centers at $(\pm\Pi, 0, 0), (0, 0, \pm\Pi)$. The saddles are connected by four heteroclinic orbits, as indicated in Figure 15.3.1.

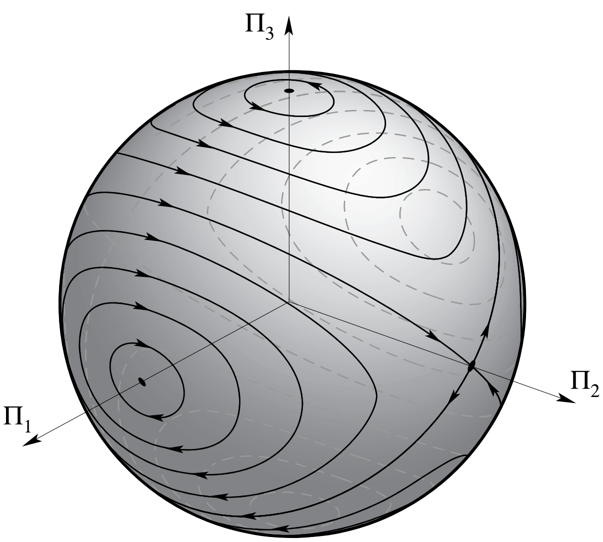

FIGURE 15.3.1. Rigid body flow on the angular momentum spheres.

In §15.10 we prove:

Theorem 15.3.1 (Rigid Body Stability Theorem). *In the motion of a free rigid body, rotation around the long and short axes are (Liapunov) stable and rotation about the middle axis is unstable.*

Even though we completely solved the rigid body equations in body representation, the actual configuration of the body, that is, its attitude in space, has not been determined yet. This will be done in §15.8. Also, one has to be careful about the meaning of stability in space versus material versus body representation.

Euler's equations are very general. The n-dimensional case has been treated by Mishchenko and Fomenko [1976, 1978a], Adler and van Moerbeke [1980a,b], and Ratiu [1980, 1981, 1982] in connection with Lie algebras and algebraic geometry. The Russian school has generalized these equations further to a large class of Lie algebras and proved their complete integrability in a long series of papers starting in 1978; see the treatise of Fomenko and Trofimov [1989] and references therein.

15.4 Kinematics on Lie Groups

We now generalize the notation used for the rigid body to any Lie group. This abstraction unifies ideas common to rigid bodies, fluids, and plasmas in a consistent way. If G is a Lie group, and $H : T^*G \to \mathbb{R}$ is a Hamiltonian, we say it is described in the ***material picture***. If $\alpha \in T_g^*G$, its ***spatial representation*** is defined by

$$\alpha^S = T_e^* R_g(\alpha), \tag{15.4.1}$$

while its ***body representation*** is

$$\alpha^B = T_e^* L_g(\alpha). \tag{15.4.2}$$

Similar notation is used for TG; if $V \in T_gG$, we get

$$V^S = T_g R_{g^{-1}}(V) \tag{15.4.3}$$

and

$$V^B = T_g L_{g^{-1}}(V). \tag{15.4.4}$$

Thus, we get body and space isomorphisms as follows:

$$\text{(Body)}\ \ G \times \mathfrak{g}^* \xleftarrow{\text{LeftTranslate}} T^*G \xrightarrow{\text{RightTranslate}} G \times \mathfrak{g}^*\ \ \text{(Space)}.$$

Thus,

$$\alpha^S = \mathrm{Ad}^*_{g^{-1}}\ \alpha^B \tag{15.4.5}$$

and

$$V^S = \mathrm{Ad}_g\ `V^B. \tag{15.4.6}$$

Part of the general theory of Chapter 13 says that if H is left (respectively, right) invariant on T^*G, it induces a Lie–Poisson system on $\mathfrak{g}^*_-$ (respectively, $\mathfrak{g}^*_+$).

Exercises

◇ **Exercise 15.4-1.** Cayley–Klein parameters Recall that the Lie algebras of SO(3) and SU(2) are the same. Recall also that SU(2) acts symplectically on $\mathbb{C}^2$ by multiplication of (complex) matrices. Use this to produce a momentum map $\mathbf{J} : \mathbb{C}^2 \to \mathrm{su}(2)^* \cong \mathbb{R}^3$.

(i) Write down $\mathbf{J}$ explicitly.

(ii) Verify by hand that $\mathbf{J}$ is a Poisson map.

(iii) If H is the rigid body Hamiltonian, compute $H_{CK} = H \circ \mathbf{J}$.

(iv) Write down Hamilton's equations for H_{CK} and discuss the collective Hamiltonian theorem in this context.

(v) Find this material, and relate it to the present context in one of the standard books (Whittaker, Pars, Hamel, or Goldstein, for example).

15.5 Poinsot's Theorem

Recall from §15.3 that the spatial angular momentum vector $\boldsymbol{\pi}$ is constant under the flow of the free rigid body. Also, if $\boldsymbol{\omega}$ is the angular velocity in space, then

$$\boldsymbol{\omega} \cdot \boldsymbol{\pi} = \boldsymbol{\Omega} \cdot \boldsymbol{\Pi} = 2K \tag{15.5.1}$$

is a constant. From this, it follows that $\boldsymbol{\omega}$ moves in an (affine) plane perpendicular to the fixed vector $\boldsymbol{\pi}$, called the ***invariable plane***. The distance from the origin to this plane is $2K/\|\boldsymbol{\pi}\|$. The ***ellipsoid of inertia in the body*** is defined by

$$\mathfrak{E} = \{\boldsymbol{\Omega} \in \mathbb{R}^3 \mid \boldsymbol{\Omega} \cdot \mathbf{I}\boldsymbol{\Omega} = 2K\}.$$

The ***ellipsoid of inertia in space*** is

$$\mathbf{R}(\mathfrak{E}) = \{\mathbf{u} \in \mathbb{R}^3 \mid \mathbf{u} \cdot \mathbf{R}\mathbf{I}\mathbf{R}^{-1}\mathbf{u} = 2K\},$$

where $\mathbf{R} = \mathbf{R}(t) \in \mathrm{SO}(3)$ denotes the configuration of the body at time t.

Theorem 15.5.1 (Poinsot's Theorem). *The moment of inertia ellipsoid in space rolls without slipping on the invariable plane.*

Proof. First, we determine the planes perpendicular to the fixed vector $\boldsymbol{\pi}$ and tangent to $\mathbf{R}(\mathfrak{E})$. See Figure 15.5.1. At the point of tangency $\mathbf{u}$, the vector $\mathbf{2RIR}^{-1}\mathbf{u}$ (the gradient of the expression defining $\mathbf{R}(\mathfrak{E})$) is proportional to $\boldsymbol{\pi}$, that is, there is an $a \in \mathbb{R}$ such that $\mathbf{R}\mathbf{I}\mathbf{R}^{-1}\mathbf{u} = a\boldsymbol{\pi}$, or

$$\mathbf{u} = a\mathbf{R}\mathbf{I}^{-1}\mathbf{R}^{-1}\boldsymbol{\pi} = a\mathbf{R}\mathbf{I}^{-1}\boldsymbol{\Pi} = a\mathbf{R}\boldsymbol{\Omega} = a\boldsymbol{\omega}$$

by (15.3.7), the definition of $\mathbf{\Pi}$, and the relation $\boldsymbol{\omega} = \mathbf{R}\mathbf{\Omega}$. However, this point $\mathbf{u} = a\boldsymbol{\omega}$ must belong to $\mathbf{R}(\mathfrak{E})$ so that using the same relations again, we get

$$2K = a^2 \boldsymbol{\omega} \cdot \mathbf{R}\mathbb{I}\mathbf{R}^{-1}\boldsymbol{\omega} = a^2 \mathbf{\Omega} \cdot \mathbb{I}\mathbf{\Omega} = 2a^2 K,$$

whence $a = \pm 1$, that is, there are exactly two planes perpendicular to $\boldsymbol{\pi}$ and tangent at $\pm\boldsymbol{\omega}$ to $\mathbf{R}(\mathfrak{E})$.

Second, we show that the plane tangent to $\mathbf{R}(\mathfrak{E})$ at $\boldsymbol{\omega}$ is the invariable plane. Indeed, since the equation of this plane is $\mathbf{u} \cdot \boldsymbol{\pi} = C$ for some constant C and $\boldsymbol{\omega}$ is in the plane, it follows that $C = \boldsymbol{\omega} \cdot \boldsymbol{\pi} = 2K$, that is, the equation of the plane is $\mathbf{u} \cdot \boldsymbol{\pi} = 2K$, which is the invariable plane.

Third, since the point of tangency is $\boldsymbol{\omega}$, which is the instantaneous axis of rotation, its velocity is zero, that is, the rolling of the inertia ellipsoid on the invariable plane takes place without slipping. ■

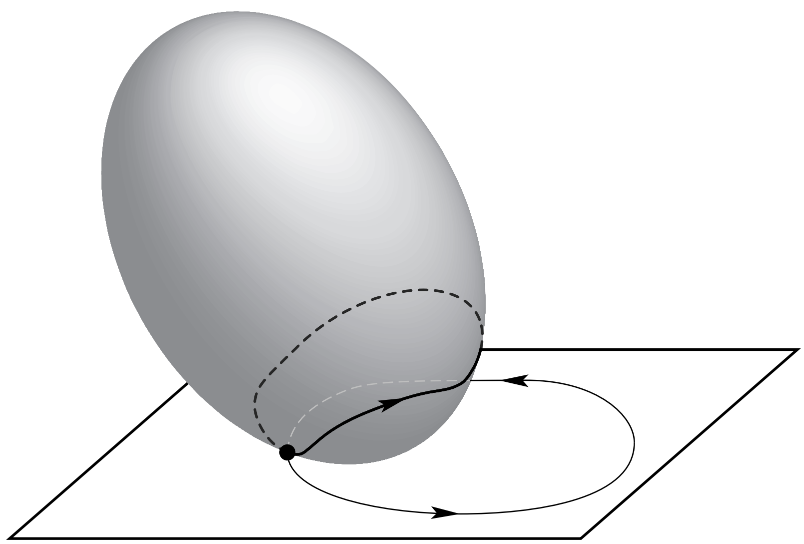

FIGURE 15.5.1. The geometry of Poinsot's theorem.

Exercises

◇ **Exercise 15.5-1.** Prove a generalization of Poinsot's theorem to any Lie algebra $\mathfrak{g}$ as follows. Assume that $l : \mathfrak{g} \to \mathbb{R}$ is a quadratic Lagrangian; that is, a map of the form

$$l(\xi) = \frac{1}{2} \langle \xi, A\xi \rangle$$

where $A : \mathfrak{g} \to \mathfrak{g}^*$ is a (symmetric) isomorphism.

Define the ***energy ellipsoid with value*** E_0 to be

$$\mathcal{E}_0 = \{\xi \in \mathfrak{g} \mid l(\xi) = E_0\}.$$

If $\xi(t)$ is a solution of the Euler–Poincaré equations and

$$g(t)^{-1}\dot{g}(t) = \xi(t),$$

with $g(0) = e$, call the set

$$\mathcal{E}_t = g(t)(\mathcal{E}_0)$$

the ***energy ellipsoid at time*** t***. Let*** $\mu = A\xi$ ***be the body momentum and***

$$\mu_s = \mathrm{Ad}^*_{g^{-1}}\, \mu$$

the conserved spatial momentum. Define the ***invariable plane*** to be the affine plane

$$\mathcal{I} = \xi(0) + \{\xi \in \mathfrak{g} \mid \langle \mu_s, \xi \rangle = 0\},$$

where $\xi(0)$ is the initial condition.

(a) Show that $\xi_s(t) = \mathrm{Ad}_{g(t)}\, \xi(t)$, the spatial velocity, lies in $\mathcal{I}$ for all t; that is, $\mathcal{I}$ is ***invariant***.

(b) Show that $\xi_s(t) \in \mathcal{E}_t$ and that the surface $\mathcal{E}_t$ is tangent to $\mathcal{I}$ at this point.

(c) Show in a precise sense that $\mathcal{E}_t$ rolls without slipping on the invariable plane.

15.6 Euler Angles

In what follows, we adopt the conventions of Arnold [1989], Cabannes [1962], Goldstein [1980], and Hamel [1949]; these are different from the ones used by the British school (Whittaker [1927] and Pars [1965]).

Let (x^1, x^2, x^3) and (χ^1, χ^2, χ^3) denote the components of a vector written in the basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ and $(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3)$, respectively. We pass from the basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ to the basis $(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3)$ by means of three consecutive counterclockwise rotations (see Figure 15.6.1). First rotate $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ by an angle φ around $\mathbf{e}_3$ and denote the resulting basis and coordinates by $(\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3)$ and (x'_1, x'_2, x'_3), respectively. The new coordinates (x'^1, x'^2, x'^3)

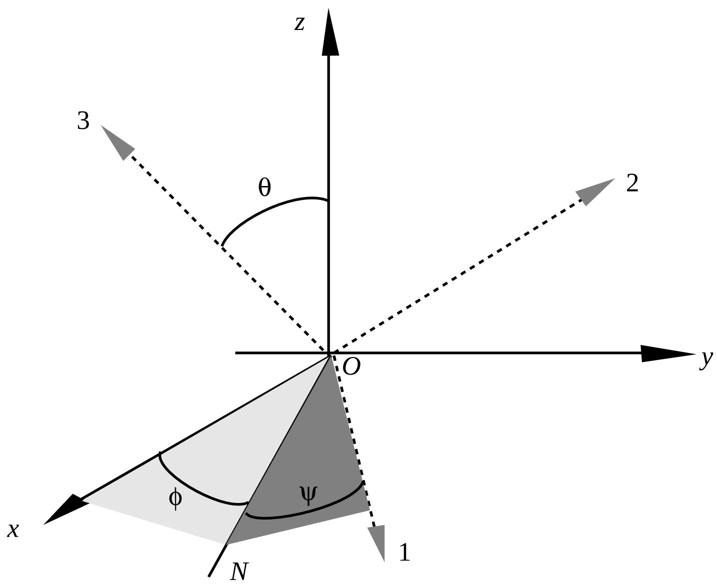

FIGURE 15.6.1. Euler angles.

are expressed in terms of the old coordinates (x^1, x^2, x^3) of the *same point* by

$$\begin{bmatrix} x'^1 \\ x'^2 \\ x'^3 \end{bmatrix} = \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \\ x^3 \end{bmatrix}. \tag{15.6.1}$$

Denote the change of basis matrix (15.6.1) in $\mathbb{R}^3$ by $\mathbf{R}_1$. Second, rotate $(\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3)$ by the angle θ around $\mathbf{e}'_1$ and denote the resulting basis and coordinate system by $(\mathbf{e}''_1, \mathbf{e}''_2, \mathbf{e}''_3)$ and (x''^1, x''^2, x''^3), respectively. The new coordinates (x''^1, x''^2, x''^3) are expressed in terms of the old coordinates (x'^1, x'^2, x'^3) by

$$\begin{bmatrix} x''^1 \\ x''^2 \\ x''^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x'^1 \\ x'^2 \\ x'^3 \end{bmatrix}. \tag{15.6.2}$$

Denote the change of basis matrix in (15.6.2) by $\mathbf{R}_2$. The $\mathbf{e}'_1$-axis, that is, the intersection of the $(\mathbf{e}_1, \mathbf{e}_2)$-plane with the $(\mathbf{e}''_1, \mathbf{e}''_2)$-plane is called the ***line of nodes*** and is denoted by ON. Finally, rotate by the angle ψ around $\mathbf{e}''_3$. The resulting basis is $(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3)$ and the new coordinates (χ^1, χ^2, χ^3) are expressed in terms of the old coordinates (x''^1, x''^2, x''^3) by

$$\begin{bmatrix} \chi^1 \\ \chi^2 \\ \chi^3 \end{bmatrix} = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x''^1 \\ x''^2 \\ x''^3 \end{bmatrix}. \tag{15.6.3}$$

Let $\mathbf{R}_3$ denote the change of basis matrix in (15.6.3). The rotation $\mathbf{R}$ sending (x^1, x^2, x^3) to (χ^1, χ^2, χ^3) is described by the matrix $\mathbf{P} = \mathbf{R}_3\mathbf{R}_2\mathbf{R}_1$ given by

$$\begin{bmatrix} \cos\psi\cos\varphi - \cos\theta\sin\varphi\sin\psi & \cos\psi\sin\varphi + \cos\theta\cos\varphi\sin\psi & \sin\theta\sin\psi \\ -\sin\psi\cos\varphi - \cos\theta\sin\varphi\cos\psi & -\sin\psi\sin\varphi + \cos\theta\cos\varphi\cos\psi & \sin\theta\cos\psi \\ \sin\theta\sin\varphi & -\sin\theta\cos\varphi & \cos\theta \end{bmatrix}.$$

Thus, $\chi = \mathbf{P}x$; equivalently, since the *same* point is expressed in two ways as $\sum_{i=1}^3 \chi^i \boldsymbol{\xi}_i = \sum_{j=1}^3 x^i \mathbf{e}_j$, we get

$$\sum_{j=1}^3 x^j \mathbf{e}_j = \sum_{i=1}^3 \chi^i \boldsymbol{\xi}_i = \sum_{i=1}^3 \left(\sum_{j=1}^3 P_{ij} x^j \right) \boldsymbol{\xi}_i = \sum_{j=1}^3 x^j \sum_{i=1}^3 P_{ij} \boldsymbol{\xi}_i,$$

that is,

$$\mathbf{e}_j = \sum_{i=1}^3 P_{ij} \boldsymbol{\xi}_i, \tag{15.6.4}$$

and hence $\mathbf{P}$ is the change of basis matrix between the rotated basis $(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3)$, and the fixed spatial basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$. On the other hand, (15.6.4) represents the matrix expression of the rotation $\mathbf{R}^T$ sending $\boldsymbol{\xi}_j$ to $\mathbf{e}_j$, that is, the matrix $[\mathbf{R}]_\xi$ of $\mathbf{R}$ in the basis $(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3)$ is P^T:

$$[\mathbf{R}]_\xi = \mathbf{P}^T, \quad \text{i.e.,} \quad \mathbf{R}\boldsymbol{\xi}_i = \sum_{i=1}^3 P_{ij} \boldsymbol{\xi}_j. \tag{15.6.5}$$

Consequently, the matrix $[\mathbf{R}]_{\mathrm{e}}$ of $\mathbf{R}$ in the basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is given by P:

$$[\mathbf{R}]_{\mathrm{e}} = \mathbf{P}, \quad \text{i.e.,} \quad \mathbf{R}\mathbf{e}_j = \sum_{i=1}^3 P_{ij} \mathbf{e}_i. \tag{15.6.6}$$

It is straightforward to check that if

$$0 \le \varphi < 2\pi, \quad 0 \le \psi < 2\pi, \quad 0 \le \theta < \pi,$$

there is a bijective map between the (φ, ψ, θ) variables and SO(3). However, this bijective map does not define a chart, since its differential vanishes, for example, at $\varphi = \psi = \theta = 0$. The differential is nonzero for

$$0 < \varphi < 2\pi, \quad 0 < \psi < 2\pi, \quad 0 < \theta < \pi,$$

and on this domain, the Euler angles do form a chart.

15.7 The Hamiltonian of the Free Rigid Body in the Material Description via Euler Angles

The Hamiltonian of the Free Rigid Body

To express the kinetic energy in terms of Euler angles, we choose the basis $\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3$ of $\mathbb{R}^3$ in the reference configuration to equal the basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ of $\mathbb{R}^3$ in the spatial coordinate system. Thus, the matrix representation of $\mathbf{R}(t)$ in the basis $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3$ equals $\mathbf{P}^T$, where $\mathbf{P}$ is given by (15.6). In this way, $\boldsymbol{\omega}$ and $\boldsymbol{\Omega}$ have the following expressions in the basis $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3$:

$$\boldsymbol{\omega} = \begin{bmatrix} \dot\theta\cos\varphi + \dot\psi\sin\varphi\sin\theta \\ \dot\theta\sin\varphi - \dot\psi\cos\varphi\sin\theta \\ \dot\varphi + \dot\psi\cos\theta \end{bmatrix}, \quad \boldsymbol{\Omega} = \begin{bmatrix} \dot\theta\cos\psi + \dot\varphi\sin\psi\sin\theta \\ -\dot\theta\sin\psi + \dot\varphi\cos\psi\sin\theta \\ \dot\varphi\cos\theta + \dot\psi \end{bmatrix}. \tag{15.7.1}$$

By definition of $\boldsymbol{\Pi}$, it follows that

$$\boldsymbol{\Pi} = \begin{bmatrix} I_1(\dot\varphi\sin\theta\sin\psi + \dot\theta\cos\psi) \\ I_2(\dot\varphi\sin\theta\cos\psi - \dot\theta\sin\psi) \\ I_3(\dot\varphi\cos\theta + \dot\psi) \end{bmatrix}. \tag{15.7.2}$$

This expresses $\boldsymbol{\Pi}$ in terms of coordinates on $T(\mathrm{SO}(3))$. Since $T(\mathrm{SO}(3))$ and $T^*(\mathrm{SO}(3))$ are to be identified by the metric defined as the left invariant metric given at the identity by $\langle\!\langle\, , \rangle\!\rangle$, the variables $(p_\varphi, p_\psi, p_\theta)$ canonically conjugate to $(\varphi, \psi.\theta)$ are given by the Legendre transformation $p_\varphi = \partial K/\partial\dot\varphi, p_\psi = \partial K/\partial\dot\psi, p_\theta = \partial K/\partial\dot\theta$, where the expression of the kinetic energy on $T(\mathrm{SO}(3))$ is obtained by plugging (15.7.2) into (15.3.3). We get

$$\begin{aligned} p_\varphi &= I_1(\dot\varphi\sin\theta\sin\psi + \dot\theta\cos\psi)\sin\theta\sin\psi \\ &\qquad + I_2(\dot\varphi\sin\theta\cos\varphi - \dot\theta\sin\psi)\sin\theta\cos\psi + I_3(\dot\varphi\cos\theta + \dot\psi)\cos\theta, \\ p_\psi &= I_3(\dot\varphi\cos\theta + \dot\psi), \\ p_\theta &= I_1(\dot\varphi\sin\theta\sin\psi + \dot\theta\cos\psi)\cos\psi \\ &\qquad - I_2(\dot\varphi\sin\theta\cos\psi - \dot\theta\sin\psi)\sin\psi, \end{aligned} \tag{15.7.3}$$

whence by (15.7.2)

$$\boldsymbol{\Pi} = \begin{bmatrix} ((p_\varphi - p_\psi\cos\theta)\sin\psi + p_\theta\sin\theta\cos\psi)/\sin\theta \\ ((p_\varphi - p_\psi cos\theta)\cos\psi - p_\theta\sin\theta\sin\psi)/\sin\theta \\ p_\psi \end{bmatrix}, \tag{15.7.4}$$

and so by (15.3.3) we get the coordinate expression of the kinetic energy in the material picture to be

$$
\begin{aligned}
K(\varphi, \psi, \theta, p_\varphi, p_\psi, p_\theta) \\
= \frac{1}{2}\Bigg\{ \frac{[(p_\varphi - p_\psi \cos\theta)\sin\psi + p_\theta \sin\theta\cos\psi]^2}{I_1 \sin^2\theta} \\
+ \frac{[(p_\varphi - p_\psi \cos\theta)\cos\psi - p_\theta \sin\theta\sin\psi]^2}{I_1 \sin^2\theta} + \frac{p_\psi^2}{I_3}\Bigg\}. \quad (15.7.5)
\end{aligned}
$$

This expression for the kinetic energy has an invariant expression on the cotangent bundle $T^*(\mathrm{SO}(3))$. In fact,

$$
K(\alpha_R) = \frac{1}{2}\langle\!\langle \boldsymbol{\Omega}, \boldsymbol{\Omega} \rangle\!\rangle = \frac{1}{4}\mathrm{Tr}(\mathbf{l}\mathbf{R}^{-1}\dot{\mathbf{R}}\mathbf{R}^{-1}\dot{\mathbf{R}}), \qquad (15.7.6)
$$

where $\alpha_{\mathbf{R}} \in T^*_{\mathbf{R}}(\mathrm{SO}(3))$ is defined by $\langle \alpha, \mathbf{R}\hat{\mathbf{v}} \rangle = \langle\!\langle \boldsymbol{\Omega}, \mathbf{v} \rangle\!\rangle$ for all $\mathbf{v} \in \mathbb{R}^3$.

The equation of motion (15.3.9) can also be derived “by hand” without appeal to Lie–Poisson or Euler–Poincaré reduction as follows. Hamilton’s canonical equations

$$
\dot{\varphi} = \frac{\partial K}{\partial p_\varphi}, \quad \dot{\psi} = \frac{\partial K}{\partial p_\psi}, \quad \dot{\theta} = \frac{\partial K}{\partial p_\theta},
$$

$$
\dot{p}_\varphi = -\frac{\partial K}{\partial \varphi}, \quad \dot{p}_\psi = -\frac{\partial K}{\partial \psi}, \quad \dot{p}_\theta = -\frac{\partial K}{\partial \theta},
$$

in a chart given by the Euler angles, become after direct substitution and a somewhat lengthy calculation,

$$
\dot{\boldsymbol{\Pi}} = \boldsymbol{\Pi} \times \boldsymbol{\Omega}.
$$

For $F, G : T^*(\mathrm{SO}(3)) \to \mathbb{R}$, that is, F, G are functions of $(\varphi, \psi, \theta, p_\varphi, p_\psi, p_\theta)$ in a chart given by Euler angles, the standard canonical Poisson bracket is

$$
\begin{aligned}
\{F, G\} = \frac{\partial F}{\partial \varphi}\frac{\partial G}{\partial p_\varphi} - \frac{\partial F}{\partial p_\varphi}\frac{\partial G}{\partial \varphi} + \frac{\partial F}{\partial \psi}\frac{\partial G}{\partial p_\psi} \\
- \frac{\partial F}{\partial p_\psi}\frac{\partial G}{\partial \psi} + \frac{\partial F}{\partial \theta}\frac{\partial G}{\partial p_\theta} - \frac{\partial F}{\partial p_\theta}\frac{\partial G}{\partial \theta}. \quad (15.7.7)
\end{aligned}
$$

A computation shows that after the substitution $(\varphi, \psi, \theta, p_\varphi, p_\psi, p_\theta) \mapsto (\Pi_1, \Pi_2, \Pi_3)$, this becomes

$$
\{F, G\}(\boldsymbol{\Pi}) = -\boldsymbol{\Pi} \cdot (\nabla F(\boldsymbol{\Pi}) \times \nabla G(\boldsymbol{\Pi})) \qquad (15.7.8)
$$

which is the $(-)$ Lie–Poisson bracket. This provides a direct check on the Lie–Poisson reduction theorem in Chapter 13. Thus (15.7.4) defines a canonical map between Poisson manifolds. The apparently “miraculous” groupings and cancellations of terms that occur in this calculation should make the reader appreciate the general theory.

Exercises

◇ **Exercise 15.7-1.** Verify that (15.7.8), namely,

$$\{F, G\}(\mathbf{\Pi}) = -\mathbf{\Pi} \cdot (\nabla F(\mathbf{\Pi}) \times \nabla G(\mathbf{\Pi}))$$

holds by a ***direct*** calculation using substitution and the chain rule.

15.8 The Analytical Solution of the Free Rigid Body Problem

We now give the analytical solution of the Euler equations. These formulae are useful when, for example, one is dealing with perturbations leading chaos via the Poincaré-Melnikov method, as in Ziglin [1980a,b], Holmes and Marsden [1983], and Koiller [1985]. For the last part of this section, the reader is assumed to be familiar with Jacobi's elementary elliptic functions; see, for example, Lawden [1989]. Let us make the following simplifying notations

$$a_1 = \frac{I_2 - I_3}{I_2 I_3} \geq 0, \quad a_2 = \frac{I_3 - I_1}{I_1 I_3} \leq 0, \quad \text{and} \quad a_3 = \frac{I_1 - I_2}{I_1 I_2} \geq 0,$$

where we assume $I_1 \geq I_2 \geq I_3 > 0$. Then Euler's equations $\dot{\mathbf{\Pi}} = \mathbf{\Pi} \times \mathbf{l}^{-1}\mathbf{\Pi}$ can be written as

$$\begin{aligned} \dot{\Pi}_1 &= a_1 \Pi_2 \Pi_3, \\ \dot{\Pi}_2 &= a_2 \Pi_3 \Pi_1, \\ \dot{\Pi}_3 &= a_3 \Pi_1 \Pi_2. \end{aligned} \tag{15.8.1}$$

For the analysis that follows it is important to recall that *the angular momentum in space is fixed* and that the instantaneous axis of rotation of the body in body coordinates is given by the angular velocity vector $\mathbf{\Omega}$.

Case 1. $I_1 = I_2 = I_3$. Then $a_1 = a_2 = a_3 = 0$ and we conclude that $\mathbf{\Pi}$, and thus $\mathbf{\Omega}$ are both constant. Hence the body rotates with constant angular velocity about a fixed axis. In Figure 15.3.1, all points on the sphere become fixed points.

Case 2. $I_1 = I_2 > I_3$. Then $a_3 = 0$ and $a_2 = -a_1$. Since $a_3 = 0$ it follows from (15.8.1) that $\Pi_3 =$ constant, and thus denoting $\lambda = -a_1 \Pi_3$ we get $a_2 \Pi_3 = \lambda$. Thus, (15.8.1) become

$$\begin{aligned} \dot{\Pi}_1 + \lambda \Pi_2 &= 0, \\ \dot{\Pi}_2 - \lambda \Pi_1 &= 0, \end{aligned}$$

which has solution for initial data given at time $t = 0$ given by

$$\begin{aligned}
\Pi_1 &= \Pi_1(0)\cos\lambda t - \Pi_2(0)\sin\lambda t,\\
\Pi_2 &= \Pi_2(0)\cos\lambda t + \Pi_1(0)\sin\lambda t.
\end{aligned}$$

These formulae say that the axis of symmetry OZ of the body rotates *relative to the body* with angular velocity λ. It is straightforward to check that OZ, $\mathbf{\Omega}$, and $\mathbf{\Pi}$ are in the same plane and that $\mathbf{\Pi}$ and $\mathbf{\Omega}$ make constant angles with OZ and thus among themselves. In addition, since $I_1 = I_2$, we have

$$\begin{aligned}
\|\mathbf{\Omega}\|^2 &= \frac{\Pi_1^2}{I_1^2} + \frac{\Pi_2^2}{I_2^2} + \frac{\Pi_3^2}{I_3^2}\\
&= \left(\frac{\Pi_1^2}{I_1} + \frac{\Pi_2^2}{I_2} + \frac{\Pi_3^2}{I_3}\right)\frac{1}{I_1} - \frac{\Pi_3^2}{I_3}\left(\frac{1}{I_1} - \frac{1}{I_3}\right)\\
&= \frac{2K}{I_1} - \frac{a_2\Pi_3^2}{I_3} = \text{constant}.
\end{aligned}$$

Therefore, the corresponding spatial objects Oz (the symmetry axis of the inertia ellipsoid in space), $\boldsymbol{\omega}$, and $\boldsymbol{\pi}$ enjoy the same properties and hence the axis of rotation in the body (given by $\mathbf{\Omega}$) makes a constant angle with the angular momentum vector that is fixed in space, and thus the axis of rotation describes a right circular cone of constant angle in space. At the same time, the axis of rotation in the body (given by $\mathbf{\Omega}$) makes a constant angle with Oz, thus tracing a second cone in the body. See Figure 15.8.1.

Consequently, the motion can be described by the rolling of a cone of constant angle in the body on a second cone of constant angle fixed in space. Whether the cone in the body rolls outside or inside the cone in space is determined by the sign of λ. Since $Oz, \boldsymbol{\omega}$, and $\boldsymbol{\pi}$ remain coplanar during the motion, $\boldsymbol{\omega}$ and Oz rotate about the fixed vector $\boldsymbol{\pi}$ with the same angular velocity, namely, the component of $\boldsymbol{\omega}$ along $\boldsymbol{\pi}$ in the decomposition of $\boldsymbol{\omega}$ relative to $\boldsymbol{\pi}$ and the Oz-axis. This angular velocity is called the ***angular velocity of precession***. Let $\mathbf{e}$ denote the unit vector along Oz and write $\boldsymbol{\omega} = \alpha\boldsymbol{\pi} + \beta\mathbf{e}$. Therefore,

$$\begin{aligned}
2K &= \boldsymbol{\omega}\cdot\boldsymbol{\pi} = \alpha\|\boldsymbol{\pi}\|^2 + \beta\mathbf{e}\cdot\boldsymbol{\pi} = \alpha\|\boldsymbol{\pi}\|^2 + \beta\Pi_3,\\
\frac{\Pi_3}{I_3} &= \mathbf{\Omega}^3 = \boldsymbol{\omega}\cdot\mathbf{e} = \alpha\boldsymbol{\pi}\cdot\mathbf{e} + \beta = \alpha\Pi_3 + \beta,
\end{aligned}$$

and

$$\beta = -a_2\Pi_3,$$

so that $\alpha = 1/I_1$ and $\beta = -a_2\Pi_3$. Therefore, the *angular velocity of precession equals* $\mathbf{\Pi}_S/I_1$.

On the $\mathbf{\Pi}$-sphere, the dynamics reduce to two fixed points surrounded by oppositely oriented periodic lines of latitude and separated by an equator of fixed points. A similar analysis applies if $I_1 > I_2 = I_3$.

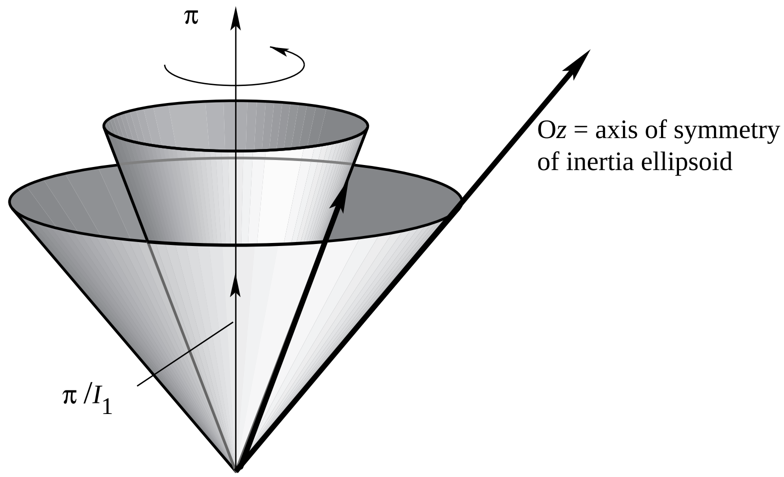

FIGURE 15.8.1. The geometry for integrating Euler's equations.

Case 3. $I_1 > I_2 > I_3$. The two integrals of energy and angular momentum

$$\frac{\Pi_1^2}{I_1} + \frac{\Pi_2^2}{I_2} + \frac{\Pi_3^3}{I_3} = 2h = ab^2, \tag{15.8.2}$$

$$\Pi_1^2 + \Pi_2^2 + \Pi_3^2 = \|\mathbf{\Pi}\|^2 = a^2b^2, \tag{15.8.3}$$

where $a = \|\mathbf{\Pi}\|^2/2h, b = 2h/\|\mathbf{\Pi}\|$ are positive constants, enable us to express Π_1 and Π_3 in terms of Π_2 as

$$\Pi_1^2 = \frac{I_1(I_2 - I_3)}{I_2(I_1 - I_3)}(\alpha^2 - \Pi_2^2) \tag{15.8.4}$$

and

$$\Pi_3^2 = \frac{I_3(I_1 - I_2)}{I_2(I_1 - I_3)}(\beta^2 - \Pi_2^2), \tag{15.8.5}$$

where α and β are positive constants given by

$$\alpha^2 = \frac{aI_2(a - I_3)b^2}{I_2 - I_3} \quad \text{and} \quad \beta^2 = \frac{aI_2(I_1 - a)b^2}{I_1 - I_2}. \tag{15.8.6}$$

By the definition of a, note that $I_1 \geq a \geq I_3$. The endpoints of the interval $[I_1, I_3]$ are easy to deal with. If $a = I_1$, then $\Pi_2 = \Pi_3 = 0$ and the motion is a steady rotation about the $\mathbf{\Pi}$-axis with body angular velocity $\pm b$. Similarly, if $a = I_3$, then $\Pi_1 = \Pi_2 = 0$. So we can assume that $I_1 > a > I_3$. With these expressions, the square of (15.8.1) becomes

$$(\dot{\Pi}_2)^2 = a_1a_3(\alpha^2 - \Pi_2^2)(\beta^2 - \Pi_2^2) \tag{15.8.7}$$

that is,

$$t = \int_{\Pi_2(0)}^{\Pi_2} \frac{du}{\sqrt{a_1 a_3(\alpha^2 - u^2)(\beta^2 - u^2)}} \tag{15.8.8}$$

which shows that Π_2, and hence Π_1, Π_3 are elliptic functions of time.

In case the quartic under the square root has double roots, that is, $\alpha = \beta$, (15.8.8) can be integrated explicitly by means of elementary functions. By (15.8.6) if follows that

$$\beta^2 - \alpha^2 = \frac{ab^2 I_2(I_1 - I_3)(I_2 - a)}{(I_1 - I_2)(I_2 - I_3)}.$$

Thus $\alpha = \beta$ if and only if $a = I_2$ which in turn forces $\alpha = \beta = ab = \|\mathbf{\Pi}\|$ and $\|\mathbf{\Pi}\|^2 = 2hI_2$. Thus (15.8.7) becomes

$$(\dot{\Pi}_2)^2 = a_1 a_3(\|\mathbf{\Pi}\|^2 - \Pi_2^2)^2. \tag{15.8.9}$$

If $\|\mathbf{\Pi}\|^2 = 2hI_2$ is satisfied, the intersection of the sphere of constant angular momentum $\|\mathbf{\Pi}\|$ with the elliptical energy surface corresponding to the value $2h$ consists of two great circles on the sphere going through the Π_2-axis in the planes

$$\Pi_3 = \pm \Pi_1 \sqrt{\frac{a_3}{a_1}}.$$

In other words, the solution of (15.8.9) consists of four heteroclinic orbits and the values $\Pi_2 = \pm\|\mathbf{\Pi}\|$. Equation (15.8.9) is solved by putting $\Pi_2 = \|\mathbf{\Pi}\| \tanh \theta$. Setting $\Pi_2(0) = 0$ for simplicity we get the four heteroclinic orbits

$$\begin{aligned}
\Pi_1^+(t) &= \pm\|\mathbf{\Pi}\| \sqrt{\frac{a_1}{-a_2}} \operatorname{sech}(-\sqrt{a_1 a_3}\, \|\mathbf{\Pi}\| t), \\
\Pi_2^+(t) &= \pm\|\mathbf{\Pi}\| \tanh(-\sqrt{a_1 a_3}\, \|\mathbf{\Pi}\,\| t), \\
\Pi_3^+(t) &= \pm\|\mathbf{\Pi}\| \sqrt{\frac{a_3}{-a_2}} \operatorname{sech}(-\sqrt{a_1 a_3}\, \|\mathbf{\Pi}\| t),
\end{aligned} \tag{15.8.10}$$

when

$$\Pi_3 = \Pi_1 \sqrt{\frac{a_3}{a_1}}$$

and

$$\Pi_1^-(t) = \Pi_1^+(-t), \quad \Pi_2^-(t) = \Pi_2^+(-t), \quad \Pi_3^-(t) = \Pi_3^+(-t),$$

when

$$\Pi_3 = -\Pi_1 \sqrt{\frac{a_3}{a_1}}.$$

If $\alpha \neq \beta$, then $a \neq I_2$, and the integration is performed with the aid of Jacobi's elliptic functions (see Whittaker and Watson [1940], Chapter 22, or Lawden [1989]). For example, the elliptic function $\operatorname{sn} u$ with modulus k is given by

where does this case end

$$\operatorname{sn} u = u - \frac{1}{3!}(1+k^2)u^3 + \frac{1}{5!}(1+14k^2+k^4)u^5 - \dots$$

and its inverse is

$$\operatorname{sn}^{-1} x = \int_0^x \frac{1}{\sqrt{(1-t^2)(1-k^2t^2)}} dt, \quad 0 \leq x \leq 1.$$

Assuming $I_1 > I_2 > a > I_3$ or, equivalently, $\alpha < \beta$, the substitution of the elliptic function $\Pi_2 = \alpha \operatorname{sn} u$ in (15.8.8) with the modulus

$$k = \alpha/\beta = \left[\frac{(I_1 - I_2)(a - I_3)}{(I_1 - a)(I_2 - I_3)}\right]^{1/2},$$

gives $\dot{u}^2 = ab^2(I_1 - a)(I_2 - I_3)/I_1 I_2 I_3 = \mu^2$. We will need the identities

$$\operatorname{cn}^2 u = 1 - \operatorname{sn}^2 u, \quad \operatorname{dn}^2 u = 1 - k^2 \operatorname{sn}^2 u, \quad \text{and} \quad \frac{d}{dx}\operatorname{sn} x = \operatorname{cn} x \operatorname{dn} x.$$

With initial condition $\Pi_2(0) = 0$, this gives

$$\Pi_2 = \alpha \operatorname{sn}(\mu t). \tag{15.8.11}$$

Thus Π_2 varies between α and $-\alpha$. Choosing the time direction appropriately, we can assume without loss of generality that $\dot{\Pi}_2(0) > 0$. Note that Π_1 vanishes when Π_2 equals $\pm\alpha$ by (15.8.4), but that Π_3^2 attains its maximal value

$$\frac{I_3(I_1 - I_2)}{I_2(I_1 - I_3)}(\beta^2 - \alpha^2) = \frac{I_3(I_2 - a)ab^2}{(I_2 - I_3)} \tag{15.8.12}$$

by (15.8.5). The minimal value of Π_3^2 occurs when $\Pi_2 = 0$, that is, it is

$$\frac{I_3(I_1 - I_2)}{I_2(I_1 - I_3)}\beta^2 = \frac{I_3(I_1 - a)ab^2}{(I_1 - I_3)} =: \delta^2, \tag{15.8.13}$$

again by (15.8.5). Thus the sign of Π_3 is constant throughout the motion. Let us assume it is positive. This hypothesis together with $\dot{\Pi}_2(0) > 0$ and $a_2 < 0$ imply that $\Pi_1(0) < 0$.

Solving for Π_1 and Π_3 from (15.8.2) and (15.8.3) and remembering that $\Pi_1(0) < 0$ gives $\Pi_1(t) = -\gamma \operatorname{cn}(\mu t), \Pi_3(t) = \delta \operatorname{dn}(\mu t)$, where δ is given by (15.8.13) and

$$\gamma^2 = \frac{I_1(a - I_3)ab^2}{(I_1 - I_3)}. \tag{15.8.14}$$

Note that $\beta > \alpha > \gamma$ and, as usual, the values of γ and δ are taken to be positive. The solution of the Euler equations is therefore

$$\Pi_1(t) = -\gamma \operatorname{cn}(\mu t), \quad \Pi_2(t) = \alpha \operatorname{sn}(\mu t), \quad \Pi_3(t) = \delta \operatorname{dn}(\mu t), \tag{15.8.15}$$

with α, γ, δ given by (15.8.6), (15.8.13), (15.8.14). If κ denotes the period invariant of Jacobi's elliptic functions then Π_1 and Π_2 have period $4\kappa/\mu$ whereas Π_3 has period $2\kappa/\mu$.

Exercises

◇ **Exercise 15.8-1.** Continue this integration process and find formulas for the attitude matrix $A(t)$ as functions of time with $A(0) =$ Identity and with given body angular momentum (or velocity).

15.9 Rigid Body Stability

Following the energy-Casimir method step by step (see the Introduction), we begin with the equations

$$\dot{\mathbf{\Pi}} = \frac{d\mathbf{\Pi}}{dt} = \mathbf{\Pi} \times \mathbf{\Omega}, \tag{15.9.1}$$

where $\mathbf{\Pi}, \mathbf{\Omega} \in \mathbb{R}^3, \mathbf{\Omega}$ is the angular velocity, and $\mathbf{\Pi}$ is the angular momentum, both viewed in the body; the relation between $\mathbf{\Pi}$ and $\mathbf{\Omega}$ is given by $\Pi_j = I_j \Omega^j,\ j = 1, 2, 3$, where $I = (I_1, I_2, I_3)$ is the diagonalized moment of inertia tensor, $I_1, I_2, I_3 > 0$. This system is Hamiltonian in the Lie–Poisson structure of $\mathbb{R}^3$ given by (15.3.8) and relative to the kinetic energy Hamiltonian

$$H(\mathbf{\Pi}) = \frac{1}{2}\mathbf{\Pi} \cdot \mathbf{\Omega} = \frac{1}{2}\sum_{i=0}^{3} \frac{\Pi_i^2}{I_i}. \tag{15.9.2}$$

Recall from (15.3.12) that for a smooth function $\Phi : \mathbb{R} \to \mathbb{R}$,

$$C_\Phi(\mathbf{\Pi}) = \Phi\left(\frac{1}{2}\|\mathbf{\Pi}\|^2\right) \tag{15.9.3}$$

is a Casimir function.

1 First Variation. We find a Casimir function C_Φ such that $H_{C_\Phi} := H + C_\Phi$ has a critical point at a given equilibrium point of (15.9.1). Such points occur when $\mathbf{\Pi}$ is parallel to $\mathbf{\Omega}$. We can assume without loss of generality, that $\mathbf{\Pi}$ and $\mathbf{\Omega}$ point in the Ox-direction. After normalizing if necessary, we can assume that the equilibrium solution is $\mathbf{\Pi}_e = (1,0,0)$. The derivative of

$$H_{C_\Phi}(\mathbf{\Pi}) = \frac{1}{2}\sum_{i=0}^{3}\frac{\Pi_i^2}{I_i} + \Phi\left(\frac{1}{2}\|\mathbf{\Pi}\|^2\right)$$

is

$$\mathbf{D}H_{C_\Phi}(\mathbf{\Pi})\cdot\delta\mathbf{\Pi} = \left(\mathbf{\Omega} + \Phi'\left(\frac{1}{2}\|\mathbf{\Pi}\|^2\right)\mathbf{\Pi}\right)\cdot\delta\mathbf{\Pi}. \tag{15.9.4}$$

This equals zero at $\mathbf{\Pi}_e = (1,0,0)$, provided that

$$\Phi'\left(\frac{1}{2}\right) = -\frac{1}{I_1}. \tag{15.9.5}$$

2 Second Variation. Using (15.9.4), the second derivative of H_{C_Φ} at the equilibrium $\mathbf{\Pi}_e = (1,0,0)$ is

$$\begin{aligned}
&\mathbf{D}^2H_{C_\Phi}(\mathbf{\Pi}_e)\cdot(\delta\mathbf{\Pi},\delta\mathbf{\Pi})\\
&\quad= \delta\mathbf{\Omega}\cdot\delta\mathbf{\Pi} + \Phi'\left(\frac{1}{2}\|\mathbf{\Pi}_e\|^2\right)\|\delta\mathbf{\Pi}\|^2 + (\mathbf{\Pi}_e\cdot\delta\mathbf{\Pi})^2\Phi''\left(\frac{1}{2}\|\mathbf{\Pi}_e\|^2\right)\\
&\quad= \sum_{i=0}^{3}\frac{(\delta\Pi_i)^2}{I_i} - \frac{\|\delta\mathbf{\Pi}\|^2}{I_1} + \Phi''\left(\frac{1}{2}\right)(\delta\Pi_1)^2\\
&\quad= \left(\frac{1}{I_2}-\frac{1}{I_1}\right)(\delta\Pi_2)^2 + \left(\frac{1}{I_3}-\frac{1}{I_1}\right)(\delta\Pi_3)^2 + \Phi''\left(\frac{1}{2}\right)(\delta\Pi_1)^2.
\end{aligned} \tag{15.9.6}$$

3 Definiteness. This quadratic form is positive-definite if and only if

$$\Phi''\left(\frac{1}{2}\right) > 0 \tag{15.9.7}$$

and

$$I_1 > I_2, \quad I_1 > I_3. \tag{15.9.8}$$

Consequently,

$$\Phi(x) = -\frac{1}{I_1}x + \left(x - \frac{1}{2}\right)^2$$

satisfies (15.9.5) and makes the second derivative of H_{C_Φ} at $(1,0,0)$ positive-definite, so *stationary rotation around the longest axis is (Liapunov) stable.*

The quadratic form is negative-definite provided

$$\Phi''\left(\frac{1}{2}\right) < 0 \tag{15.9.9}$$

and

$$I_1 < I_2, \quad I_1 < I_3. \tag{15.9.10}$$

It is obvious that we may find a function Φ satisfying the requirements (15.9.5) and (15.9.9); for example, $\Phi(x) = -(1/I_1)x - \left(x - \frac{1}{2}\right)^2$. This proves that *rotation around the short axis is (Liapunov) stable*.

Finally, the quadratic form (15.9.6) is indefinite if

$$I_1 > I_2, \quad I_3 > I_1, \tag{15.9.11}$$

or the other way around. We cannot show by this method that rotation around the middle axis is *unstable*. We shall prove, by using a spectral analysis, that rotation about the middle axis is, in fact, unstable. Linearizing (15.9.1) at $\mathbf{\Pi}_e = (1, 0, 0)$ yields the linear constant coefficient system

$$\begin{aligned}
(\delta\dot{\Pi}) &= \delta\Pi \times \mathbf{\Omega}_e + \Pi_e \times \delta\mathbf{\Omega} \\
&= \left(0, \frac{I_3 - I_1}{I_3 I_1}\delta\Pi_3, \frac{I_1 - I_2}{I_1 I_2}\delta\Pi_2\right) \\
&= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \dfrac{I_3 - I_1}{I_3 I_1} \\ 0 & \dfrac{I_1 - I_2}{I_1 I_2} & 0 \end{bmatrix} \delta\Pi.
\end{aligned} \tag{15.9.12}$$

On the tangent space at $\mathbf{\Pi}_e$ to the sphere of radius $\|\mathbf{\Pi}_e\| = 1$, the linear operator given by this linearized vector field has a matrix given by the lower right (2×2)-block whose eigenvalues are

$$\pm \frac{1}{I_1\sqrt{I_2 I_3}} \sqrt{(I_1 - I_2)(I_3 - I_1)}.$$

Both of them are real by (15.9.11) and one is strictly positive. Thus $\mathbf{\Pi}_e$ is spectrally unstable and thus is unstable.

We summarize the results in the following theorem.

Theorem 15.9.1 (Rigid Body Stability Theorem). *In the motion of a free rigid body, rotation around the long and short axes is (Liapunov) stable and around the middle axis is unstable.*

It is important to keep the Casimir functions as general as possible, because otherwise (15.9.5) and (15.9.9) could be contradictory. Had we simply chosen

$$\Phi(x) = -\frac{1}{I_1}x + \left(x - \frac{1}{2}\right)^2,$$

(15.9.5) would be verified, but (15.9.9) would not. It is only the choice of *two different* Casimirs that enables us to prove the two stability results, even though the level surfaces of these Casimirs are the same.

Remarks.

1. As we have seen, rotations about the intermediate axis are unstable and this is even for the linearized equations. The unstable homoclinic orbit that connect the two unstable points have interesting features. Not only are they interesting because of the chaotic solutions via the Poincaré-Melnikov method that can be obtained in various perturbed systems (see Holmes and Marsden [1983], Wiggins [1988], and references therein), but already, the orbit itself is interesting since a rigid body tossed about its middle axis will undergo an interesting half twist when the opposite saddle point is reached, even though the rotation axis has returned to where it was. The reader can easily perform the experiment; see Ashbaugh, Chicone, and Cushman [1990] and Montgomery [1991a] for more information.

2. The same stability theorem can also be proved by working with the second derivative along a coadjoint orbit in $\mathbb{R}^3$; that is, a two-sphere; see Arnold [1966a]. This coadjoint orbit method also *suggests* instability of rotation around the intermediate axis.

3. Dynamic stability on the Π-sphere has been shown. What about the stability of the dynamically rigid body we "see"? This can be deduced from what we have done. Probably the best approach though is to use the relation between the reduced and unreduced dynamics; see Simo, Lewis, and Marsden [1991] and Lewis [1992] for more information.

4. When the body angular momentum undergoes a periodic motion, the actual motion of the rigid body in space is not periodic. In the introduction we described the associated geometric phase.

5. See Lewis and Simo [1990] and Simo, Lewis, and Marsden [1991] for related work on deformable elastic bodies (pseudo-rigid bodies). ♦

Exercises

◇ **Exercise 15.9-1.** Let $\mathbf{B}$ be a given fixed vector in $\mathbb{R}^3$ and let M evolve by $\dot{M} = M \times B$. Show that this evolution is Hamiltonian. Determine the equilibria and their stability.

◇ **Exercise 15.9-2.** Consider the following modification of the Euler equations:

$$\dot{\mathbf{\Pi}} = \mathbf{\Pi} \times \Omega + \alpha \mathbf{\Pi} \times (\mathbf{\Pi} \times \Omega),$$

where α is a positive constant. Show that,

(a) The spheres $\|\mathbf{\Pi}\|^2$ are preserved.

(b) Energy is strictly decreasing except at equilibria.

(c) The equations can be written in the form

$$\dot{F} = \{F, H\}_{\mathrm{rb}} + \{F, H\}_{\mathrm{sym}},$$

where the first bracket is the usual rigid body bracket and the second is the *symmetric* bracket

$$\{F, K\}_{\mathrm{sym}} = \alpha(\Pi \times \nabla F) \cdot (\Pi \times \nabla K).$$

15.10 Heavy Top Stability

The heavy top equations are

$$\frac{d\mathbf{\Pi}}{dt} = \mathbf{\Pi} \times \mathbf{\Omega} + Mgl\mathbf{\Gamma} \times \boldsymbol{\chi}, \tag{15.10.1}$$

$$\frac{d\mathbf{\Gamma}}{dt} = \mathbf{\Gamma} \times \mathbf{\Omega}, \tag{15.10.2}$$

where $\mathbf{\Pi}, \mathbf{\Gamma}, \boldsymbol{\chi} \in \mathbb{R}^3$. Here $\mathbf{\Pi}$ and $\mathbf{\Omega}$ are the angular momentum and angular velocity in the body, $\Pi_i = I_i \Omega^i, I_i > 0,\ i = 1, 2, 3$, with $I = (I_1, I_2, I_3)$ the moment of inertia tensor. The vector $\mathbf{\Gamma}$ represents the motion of the unit vector along the Oz-axis as seen from the body, and the constant vector $\boldsymbol{\chi}$ is the unit vector along the line segment of length l connecting the fixed point to the center mass of the body; M is the total mass of the body, and g is the strength of the gravitational acceleration, which is along Oz pointing down.

This system is Hamiltonian in the Lie–Poisson structure of $\mathbb{R}^3 \times \mathbb{R}^3$ given in the Introduction relative to the heavy top Hamiltonian

$$H(\mathbf{\Pi}, \mathbf{\Gamma}) = \frac{1}{2}\mathbf{\Pi} \cdot \mathbf{\Omega} + Mgl\mathbf{\Gamma} \cdot \boldsymbol{\chi}. \tag{15.10.3}$$

The Poisson structure (with $\|T\| = 1$ imposed) foreshadows that of $T^* \mathrm{SO}(3)/S^1$, where S^1 acts by rotation about the axis of gravity. The fact that one gets the Lie–Poisson bracket for a semi-direct product Lie algebra is a special case of the general theory of reduction and semi-direct products (Marsden, Ratiu and Weinstein [1984a,b])

The functions $\mathbf{\Pi} \cdot \mathbf{\Gamma}$ and $\|\mathbf{\Gamma}\|^2$ are Casimir functions, as is

$$C(\mathbf{\Pi}, \mathbf{\Gamma}) = \Phi(\mathbf{\Pi} \cdot \mathbf{\Gamma}, \|\mathbf{\Gamma}\|^2), \tag{15.10.4}$$

where Φ is any smooth function from $\mathbb{R}^2$ to $\mathbb{R}$.

We shall be concerned here with the Lagrange top. This is a heavy top for which $I_1 = I_2$, that is, it is symmetric, and the center of mass lies on the axis of symmetry in the body, that is, $\boldsymbol{\chi} = (0,0,1)$. This assumption simplifies the equations of motion (15.10.1) to

$$\begin{aligned}
\dot{\Pi}_1 &= \frac{I_2 - I_3}{I_2 I_3}\Pi_2\Pi_3 + Mgl\Gamma_2, \\
\dot{\Pi}_2 &= \frac{I_3 - I_1}{I_1 I_3}\Pi_1\Pi_3 - Mgl\Gamma_1, \\
\dot{\Pi}_3 &= \frac{I_1 - I_2}{I_1 I_2}\Pi_1\Pi_2.
\end{aligned}$$

Since $I_1 = I_2$, we have $\dot{\Pi}_3 = 0$; thus Π_3 and hence any function $\varphi(\Pi_3)$ of Π_3 is conserved.

1 First Variation. We shall study the equilibrium solution

$$\boldsymbol{\Pi}_e = (0,0,\Pi_3^0), \quad \boldsymbol{\Gamma}_e = (0,0,1),$$

where $\Pi_3^0 \neq 0$, which represents the spinning of a symmetric top in its upright position. To begin, we consider conserved quantities of the form $H_{\Phi,\varphi} = H + \Phi(\boldsymbol{\Pi}\cdot\boldsymbol{\Gamma}, \|\boldsymbol{\Gamma}\|^2) + \varphi(\Pi_3)$ and which have a critical point at the equilibrium. The first derivative of $H_{\Phi,\varphi}$ is given by

$$\begin{aligned}
\mathbf{D}H_{\Phi,\varphi}(\boldsymbol{\Pi},\boldsymbol{\Gamma})\cdot(\delta\boldsymbol{\Pi},\delta\Gamma) = (\boldsymbol{\Omega} + \dot{\Phi}(\boldsymbol{\Pi}\cdot\boldsymbol{\Gamma}, \|\boldsymbol{\Gamma}\|^2)\boldsymbol{\Gamma})\cdot\delta\boldsymbol{\Pi} \\
+ [Mgl\boldsymbol{\chi} + \dot{\Phi}(\boldsymbol{\Pi}\cdot\boldsymbol{\Gamma}, \|\boldsymbol{\Gamma}\|^2)\boldsymbol{\Pi} \\
+ 2\Phi'(\boldsymbol{\Pi}\cdot\boldsymbol{\Gamma}, \|\boldsymbol{\Gamma}\|^2)\boldsymbol{\Gamma}]\cdot\delta\boldsymbol{\Gamma} + \varphi'(\Pi_3)\delta\Pi_3,
\end{aligned}$$

where $\dot{\Phi} = \partial\Phi/\partial(\boldsymbol{\Pi}\cdot\boldsymbol{\Gamma})$ and $\Phi' = \partial\Phi/\partial(\|\boldsymbol{\Gamma}\|^2)$. At the equilibrium solution $(\boldsymbol{\Pi}_e, \boldsymbol{\Gamma}_e)$ the first derivative of $H_{\Phi,\varphi}$ vanishes, provided that

$$\frac{\Pi_3^0}{I_3} + \dot{\Phi}(\Pi_3^0, 1) + \varphi'(\Pi_3^0) = 0$$

and that

$$Mgl + \dot{\Phi}(\Pi_3^0, 1)\Pi_3^0 + 2\Phi'(\Pi_3^0, 1) = 0;$$

the remaining equations, involving indices 1 and 2, are trivially verified. Solving for $\dot{\Phi}(\Pi_3^0, 1)$ and $\Phi'(\Pi_3^0, 1)$ we get the conditions

$$\dot{\Phi}(\Pi_3^0, 1) = -\left(\frac{1}{I_3} + \frac{\varphi'(\Pi_3^0)}{\Pi_3^0}\right)\Pi_3^0, \tag{15.10.5}$$

$$\Phi'(\Pi_3^0, 1) = \frac{1}{2}\left(\frac{1}{I_3} + \frac{\varphi'(\Pi_3^0)}{\Pi_3^0}\right)(\Pi_3^0)^2 - \frac{1}{2}Mgl. \tag{15.10.6}$$

2 Second Variation. We shall check for definiteness of the second variation of $H_{\Phi,\varphi}$ at the equilibrium point $(\mathbf{\Pi}_e, \mathbf{\Gamma}_e)$. To simplify the notation we shall set

$$a = \varphi''(\Pi_3^0), \qquad b = 4\Phi''(\Pi_3^0, 1), \qquad c = \ddot{\Phi}(\Pi_3^0, 1), \qquad d = 2\dot{\Phi}'(\Pi_3^0, 1).$$

With this notation, (15.10.5) and (15.10.6), we find that the matrix of the second derivative at (Π_e, Γ_e) is

$$\begin{bmatrix} 1/I_1 & 0 & 0 & \dot{\Phi}(\Pi_3^0,1) & 0 & 0 \\ 0 & 1/I_1 & 0 & 0 & \dot{\Phi}(\Pi_3^0,1) & 0 \\ 0 & 0 & (1/I_3)+a+c & 0 & 0 & a_{36} \\ \dot{\Phi}(\Pi_3^0,1) & 0 & 0 & 2\Phi'(\Pi_3^0,1) & 0 & 0 \\ 0 & \dot{\Phi}(\Pi_3^0,1) & 0 & 0 & 2\Phi'(\Pi_3^0,1) & 0 \\ 0 & 0 & a_{36} & 0 & 0 & a_{66} \end{bmatrix}, \tag{15.10.7}$$

where $a_{36} = \dot{\Phi}(\Pi_3^0, 1) + \Pi_3^0 c + d$ and $a_{66} = 2\Phi'(\Pi_3^0, 1) + b + (\Pi_3^0)^2 c + \Pi_3^0 d$.

3 Definiteness. The computations for this part will be done using the following formula from linear algebra. If

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

is a $(p+q) \times (p+q)$ matrix and if the $(p \times p)$-matrix A is invertible, then

$$\det M = \det A \det(D - CA^{-1}B).$$

If the quadratic form given by (15.10.7) is definite, it must be positive-definite since the $(1,1)$-entry is positive. Recalling that $I_1 = I_2$, the six principal determinants have the following values:

$$\left(\frac{2}{I_1}\Phi'(\Pi_3^0,1) - \dot{\Phi}(\Pi_3^0,1)^2\right)^2$$
$$\left(\dot{\Phi}(\Pi_3^0,1) + \Pi_3^0 c + d\right)^2 \left(2\Phi'\left(\Pi_3^0,1\right) + b + \left(\Pi_3^0\right)^2 c + \Pi_3^0 d\right).$$

Consequently, the quadratic form given by (15.10.7) is positive-definite, if and only if

$$\frac{1}{I_3} + a + c > 0, \tag{15.10.8}$$

$$\frac{2}{I_1}\Phi'(\Pi_3^0,1) - \dot{\Phi}(\Pi_3^0,1)^2 > 0, \tag{15.10.9}$$

and

$$2\Phi'(\Pi_3^0, 1) + b + (\Pi_3^0)^2 c + \Pi_3^0 d > 0. \tag{15.10.10}$$

Conditions (15.10.8) and (15.10.10) can always be satisfied if we choose the numbers $a, b, c,$ and d appropriately; for example, $a = c = d = 0$ and b sufficiently large and positive. Thus, the determining condition for stability is (15.10.9). By (15.10.5) and (15.10.6), this becomes

$$\frac{1}{I_1}\left[\left(\frac{1}{I_3} + \frac{\varphi'(\Pi_3^0)}{\Pi_3^0}\right)(\Pi_3^0)^2 - Mgl\right] - \left(\frac{1}{I_3} + \frac{\varphi'(\Pi_3^0)}{\Pi_3^0}\right)^2 (\Pi_3^0)^2 > 0. \tag{15.10.11}$$

We can choose $\varphi'(\Pi_3^0)$ so that

$$\frac{1}{I_3} + \frac{\varphi'(\Pi_3^0)}{\Pi_3^0} = e$$

has any value we wish. The left side of (15.10.11) is a quadratic polynomial in e, whose leading coefficient is negative. In order for this to be positive for some e, it is necessary and sufficient for the discriminant

$$\frac{(\Pi_3^0)^4}{I_1^2} - \frac{4(\Pi_3^0)^2 Mgl}{I_1}$$

to be positive; that is,

$$(\Pi_3^0)^2 > 4MglI_1$$

which is the classical stability condition for a fast top. We have proved the first part of the following:

Theorem 15.10.1 (Heavy Top Stability Theorem). *An upright spinning Lagrange top is stable provided that the angular velocity is strictly larger than $2\sqrt{MglI_1}/I_3$. It is unstable if the angular velocity is smaller than this value.*

The second part of the theorem is proved, as in §15.9, by a spectral analysis of the linearized equations, namely

$$(\delta\dot{\mathbf{\Pi}}) = \delta\mathbf{\Pi} \times \mathbf{\Omega} + \mathbf{\Pi}_e \times \delta\mathbf{\Omega} + Mgl\delta\mathbf{\Gamma} \times \boldsymbol{\chi}, \tag{15.10.12}$$

$$(\delta\dot{\mathbf{\Gamma}}) = \delta\mathbf{\Gamma} \times \mathbf{\Omega} + \mathbf{\Gamma}_e \times \delta\mathbf{\Omega}, \tag{15.10.13}$$

on the tangent space to the coadjoint orbit in $\mathfrak{se}(3)^*$ through $(\mathbf{\Pi}_e, \mathbf{\Gamma}_e)$ given by

$$\{(\delta\mathbf{\Pi}, \delta\mathbf{\Gamma}) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid \delta\mathbf{\Pi} \cdot \mathbf{\Gamma}_e + \mathbf{\Pi}_e \cdot \delta\mathbf{\Gamma} = 0 \quad \text{and} \quad \delta\mathbf{\Gamma} \cdot \mathbf{\Gamma}_e = 0\}$$

$$\cong \{(\delta\mathbf{\Pi}_1, \delta\mathbf{\Pi}_2, \delta\mathbf{\Gamma}_1, \delta\mathbf{\Gamma}_2)\} = \mathbb{R}^4. \tag{15.10.14}$$

The matrix of the linearized system of equations on this space is computed to be

$$\begin{bmatrix} 0 & \dfrac{\Pi_3^0}{I_3}\dfrac{I_1 - I_3}{I_1} & 0 & Mgl \\ -\dfrac{\Pi_3^0}{I_3}\dfrac{I_1 - I_3}{I_1} & 0 & -Mgl & 0 \\ 0 & -\dfrac{1}{I_1} & 0 & \dfrac{\Pi_3^0}{I_3} \\ \dfrac{1}{I_1} & 0 & -\dfrac{\Pi_3^0}{I_3} & 0 \end{bmatrix}. \tag{15.10.15}$$

The matrix (15.10.15) has characteristic polynomial

$$\lambda^4 + \frac{1}{I_1^2}\left[(I_1^2 + (I_1 - I_3)^2)\left(\frac{\Pi_3^0}{I_3}\right)^2 - 2MglI_1\right]\lambda^2 + \frac{1}{I_1^2}\left[(I_1 - I_3)\left(\frac{\Pi_3^0}{I_3}\right)^2 + Mgl\right]^2, \tag{15.10.16}$$

whose discriminant as a quadratic polynomial in λ^2 is

$$\frac{1}{I_1^4}(2I_1 - I_3)^2\left(\frac{\Pi_3^0}{I_3}\right)^2\left(I_3^2\left(\frac{\Pi_3^0}{I_3}\right)^2 - 4MglI_1\right).$$

This discriminant is negative if and only if

$$\Pi_3^0 < 2\sqrt{MglI_1}.$$

Under this condition the four roots of the characteristic polynomial are all distinct and equal to $\lambda_0, \bar{\lambda}_0, -\lambda_0, -\bar{\lambda}_0$ for some $\lambda_0 \in \mathbb{C}$, where $\operatorname{Re}\lambda_0 \neq 0$ and $\operatorname{Im}\lambda_0 \neq 0$. Thus, at least two of these roots have real part strictly larger than zero thereby showing that $(\mathbf{\Pi}_e, \mathbf{\Gamma}_e)$ is spectrally unstable and hence unstable.

When $I_2 = I_1 + \epsilon$ for small ϵ, the conserved quantity $\varphi(\Pi_3)$ is no longer available. In this case, a sufficiently fast top is still linearly stable, and nonlinear stability can be assessed by KAM theory. Other regions of phase space are known to possess chaotic dynamics in this case (Holmes and Marsden [1983]). For more information on stability and bifurcation in the heavy top, we refer to Lewis, Ratiu, Simo, and Marsden [1992].

Exercises

◇ **Exercise 15.10-1.** (a) Show that $\tilde{H}(\mathbf{\Pi}, \mathbf{\Gamma}) = H(\mathbf{\Pi}, \mathbf{\Gamma}) + \|\mathbf{\Gamma}\|^2/2$, where H is given by (15.10.3), generates the same equations of motion (15.10.1) and (15.10.2).

(b) Taking the Legendre transform of $\tilde{H}$, show that the equations can be written in Euler–Poincaré form.

15.11 The Rigid Body and the Pendulum

This section, following Holm and Marsden [1991], shows how the rigid body and the pendulum are linked.

Euler's equations are expressible in vector form as

$$\frac{d}{dt}\mathbf{\Pi} = \nabla H \times \nabla L, \tag{15.11.1}$$

where H is the energy,

$$H = \frac{\Pi_1^2}{2I_1} + \frac{\Pi_2^2}{2I_2} + \frac{\Pi_3^2}{2I_3}, \tag{15.11.2}$$

$$\nabla H = \left(\frac{\partial H}{\partial \Pi_1}, \frac{\partial H}{\partial \Pi_2}, \frac{\partial H}{\partial \Pi_3} \right) = \left(\frac{\Pi_1}{I_1}, \frac{\Pi_2}{I_2}, \frac{\Pi_3}{I_3} \right), \tag{15.11.3}$$

is the gradient of H and L is the square of the body angular momentum,

$$L = \frac{1}{2}\left(\Pi_1^2 + \Pi_2^2 + \Pi_3^2 \right). \tag{15.11.4}$$

Since both H and L are conserved, the rigid body motion itself takes place, as we know, along the intersections of the level surfaces of the energy (ellipsoids) and the angular momentum (spheres) in $\mathbb{R}^3$. The centers of the energy ellipsoids and the angular momentum spheres coincide. This, along with the $(\mathbb{Z}_2)^3$ symmetry of the energy ellipsoids, implies that the two sets of level surfaces in $\mathbb{R}^3$ develop collinear gradients (for example, tangencies) at pairs of points which are diametrically opposite on an angular momentum sphere. At these points, collinearity of the gradients of H and L implies stationary rotations, that is, equilibria.

Euler's equations for the rigid body may also be written as

$$\frac{d}{dt}\mathbf{\Pi} = \nabla K \times \nabla N, \tag{15.11.5}$$

where K and N are linear combinations of energy and angular momentum of the form

$$\begin{pmatrix} K \\ N \end{pmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{pmatrix} H \\ L \end{pmatrix}, \tag{15.11.6}$$

with real constants a, b, c, and d satisfying $ad - bc = 1$. Indeed, since

$$K = \frac{1}{2}\left(\frac{a}{I_1} + b\right)\pi_1^2 + \frac{1}{2}\left(\frac{a}{I_2} + b\right)\pi_3^2 + \frac{1}{2}\left(\frac{a}{I_3} + b\right)\pi_3^2 +$$

if $I_1 = I_2 = I_3$, the choice $a = -bI_1$ makes $K = 0$, that is, the diagonal from of K is $(0, 0, 0)$. If, $I_1 \neq I_2 = I_3$, the choice $a = -bI_2$ yields

$$K = \frac{b}{2}\left(1 - \frac{I_2}{I_1}\right)\Pi_1^2,$$

so the diagonal form of K is $(*, 0, 0)$. Finally, if $I_1 < I_2 < I_3$, the choice

$$a = 1, \quad b = -\frac{1}{I_3}, \quad c = -\frac{I_1 I_3}{I_3 - I_1} < 0, \quad \text{and} \quad d = \frac{I_3}{I_3 - I_1} < 0 \tag{15.11.7}$$

gives

$$K = \frac{1}{2}\left(\frac{1}{I_1} - \frac{1}{I_3}\right)\Pi_1^2 + \frac{1}{2}\left(\frac{1}{I_2} - \frac{1}{I_3}\right)\Pi_2^2 \tag{15.11.8}$$

and

$$N = \frac{I_3(I_2 - I_1)}{2I_2(I_3 - I_1)}\Pi_2^2 + \frac{1}{2}\Pi_3^2. \tag{15.11.9}$$

With this choice, the orbits for Euler's equations for rigid body dynamics are realized as motion along the intersections of two, orthogonally oriented, *elliptic cylinders,* one elliptic cylinder being a level surface of K, with its translation axis along Π_3 (where $K = 0$), and the other a level surface of N, with its translation axis along Π_1 (where $N = 0$).

For a general choice of K and N, equilibria occur at points where the gradients of K and N are collinear. This can occur at points where the level sets are tangent (and the gradients both are nonzero), or at points where one of the gradients vanishes. In the elliptic cylinder case above, these two cases are points where the elliptic cylinders are tangent, and at points where the axis of one cylinder punctures normally through the surface of the other. The elliptic cylinders are tangent at one $\mathbb{Z}_2$-symmetric pair of points along the Π_2 axis, and the elliptic cylinders have normal axial punctures at two other $\mathbb{Z}_2$-symmetric pairs of points along the Π_1 and Π_3 axes.

Let us pursue the elliptic cylinders point of view further. We now change variables in the rigid body equations within a level surface of K. To simplify notation, we first define the three positive constants $k_i^2, i = 1, 2, 3$, by setting

$$K = \frac{\Pi_1^2}{2k_1^2} + \frac{\Pi_2^2}{2k_2^2} \quad \text{and} \quad N = \frac{\Pi_2^2}{2k_3^2} + \frac{1}{2}\Pi_3^2. \tag{15.11.10}$$

For

$$\frac{1}{k_1^2} = \frac{1}{I_1} - \frac{1}{I_3}, \quad \frac{1}{k_2^2} = \frac{1}{I_2} - \frac{1}{I_3}, \quad \frac{1}{k_3^2} = \frac{I_3(I_2 - I_1)}{I_2(I_3 - I_1)}. \tag{15.11.11}$$

On the surface $K =$ constant, and setting $r = \sqrt{2K} =$ constant, define new variables θ and p by

$$\Pi_1 = k_1 r \cos\theta, \quad \Pi_2 = k_2 r \sin\theta, \quad \Pi_3 = p. \tag{15.11.12}$$

In terms of these variables, the constants of the motion become

$$K = \frac{1}{2} r^2 \quad \text{and} \quad N = \frac{1}{2} p^2 + \left(\frac{k_2^2}{2k_3^2} r^2 \right) \sin^2\theta. \tag{15.11.13}$$

As we shall show below, using a Poisson structure relevant to the equations of motion in the form $\frac{d}{dt}\mathbf{\Pi} = \nabla K \times \nabla N$, the variables θ and p are, up to a scale factor, canonically conjugate, that is, the Poisson bracket of two functions of θ and p are given in standard canonical form (up to a scale factor) as follows:

$$\{F, G\}_{\text{EllipCyl}} = \frac{1}{k_1 k_2} \left(\frac{\partial F}{\partial p} \frac{\partial G}{\partial \theta} - \frac{\partial F}{\partial \theta} \frac{\partial G}{\partial p} \right). \tag{15.11.14}$$

In particular,

$$\{p, \theta\}_{\text{EllipCyl}} = \frac{1}{k_1 k_2}. \tag{15.11.15}$$

The quantity N is the Hamiltonian in these variables—note that N has the form of kinetic plus potential energy—and the equations of motion express themselves in Hamiltonian form in terms of the canonical Poisson bracket. Namely,

$$\frac{d}{dt}\theta = \{N, \theta\}_{\text{EllipCyl}} = \frac{1}{k_1 k_2} \frac{\partial N}{\partial p} = \frac{1}{k_1 k_2} p, \tag{15.11.16}$$

$$\frac{d}{dt}p = \{N, p\}_{\text{EllipCyl}} = \frac{-1}{k_1 k_2} \frac{\partial N}{\partial \theta} = \frac{-1}{k_1 k_2} \frac{k_2^2}{k_3^2} r^2 \sin\theta\cos\theta. \tag{15.11.17}$$

Combining these equations of motion gives

$$\frac{d^2}{dt^2}\theta = \frac{-r^2}{2k_1^2 k_3^2} \sin 2\theta, \tag{15.11.18}$$

or, in terms of the original rigid body parameters,

$$\frac{d^2}{dt^2}\theta = -K \left(\frac{1}{I_1} - \frac{1}{I_2} \right) \sin 2\theta. \tag{15.11.19}$$

Thus, we have proved

Proposition 15.11.1. *Rigid body motion reduces to pendulum motion on level surfaces of K.*

Another way of saying this is as follows: regard rigid body angular momentum space as the union of the level surfaces of K, so the dynamics of the rigid body is recovered by looking at the dynamics on each of these level surfaces. On each level surface, the dynamics is equivalent to a simple pendulum. In this sense, we have proved:

Corollary 15.11.2. *The dynamics of a rigid body in three-dimensional body angular momentum space is a union of two-dimensional simple pendula phase portraits.*

By restricting to a nonzero level surface of K, the pair of rigid body equilibria along the Π_3 axis are excluded. (This pair of equilibria can be included by permuting the indices of the moments of inertia.) The other two pairs of equilibria, along the Π_1 and Π_2 axes, lie in the $p = 0$ plane at $\theta = 0, \pi/2, \pi$, and $3\pi/2$. Since K is positive, the stability of each equilibrium point is determined by the relative sizes of the principal moments of inertia, which affect the overall sign of the right-hand side of the pendulum equation. The well-known results about stability of equilibrium rotations along the least and greatest principal axes, and instability around the intermediate axis, are immediately recovered from this overall sign, combined with the stability properties of the pendulum equilibria. For $K > 0$ and $I_1 < I_2 < I_3$, this overall sign is negative, so the equilibria at $\theta = 0$ and π (along the Π_1 axis) are stable, while those at $\theta = \pi/2$ and $3\pi/2$ (along the Π_2 axis) are unstable. The factor of 2 in the argument of the sine in the pendulum equation is explained by the $\mathbb{Z}_2$ symmetry of the level surfaces of K (or, just as well, by their invariance under $\theta \mapsto \theta + \pi$). Under this discrete symmetry operation, the equilibria at $\theta = 0$ and $\pi/2$ exchange with their counterparts at $\theta = \pi$ and $3\pi/2$, respectively, while the elliptical level surface of K is left invariant. By construction, the Hamiltonian N in the reduced variables θ and p is also invariant under this discrete symmetry.

Let us return to the derivation of the Poisson bracket (15.11.4) on the level surface $K =$ constant. Recall that the *rigid body Poisson bracket* on two functions F_1 and F_2 of $\mathbf{\Pi}$ is given by the *minus Lie–Poisson bracket* for $\mathfrak{so}(3)^*$:

$$\{F_1, F_2\} = -\mathbf{\Pi} \cdot (\nabla F_1 \times \nabla F_2). \tag{15.11.20}$$

If Euler's equations are rewritten as

$$\frac{d}{dt}\mathbf{\Pi} = \nabla K \times \nabla N,$$

where K and N are given as above by an $\mathrm{SL}(2, \mathbb{R})$ matrix

$$\begin{pmatrix} K \\ N \end{pmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{pmatrix} H \\ L \end{pmatrix},$$

one checks that the equations are Hamiltonian with energy N and the Poisson bracket

$$\{F_1, F_2\}_K = -\nabla K \cdot (\nabla F_1 \times \nabla F_2). \tag{15.11.21}$$

As we saw in Exercise 1.3-2, this defines a Poisson structure and K is a Casimir function for this bracket. One can now directly verify the formula $\{F, G\}_{\text{EllipCyl}}$ for the Poisson bracket on level sets of the function K in the elliptic cylinder case by a straightforward calculation.

The rigid body can, correspondingly, be regarded as a left invariant system on the group $O(K)$ or SE(2). The special case of SE(2) is the one in which the orbits are cotangent bundles. The fact that one gets a cotangent bundle in this situation is a special case of the cotangent bundle reduction theorem (Volume II) using the semidirect product reduction theorem; see Marsden, Ratiu, and Weinstein [1984a,b]. For the Euclidean group it says that the coadjoint orbits of the Euclidean group of the plane are given by reducing the cotangent bundle of the rotation group of the plane by the trivial group, giving the cotangent bundle of a circle with its canonical symplectic structure up to a factor. This is the abstract explanation of why, in the elliptic cylinder case above, the variables θ and p were, up to a factor, canonically conjugate. This general theory is also consistent with the fact that the Hamiltonian N is of the form kinetic plus potential energy. In fact, in the cotangent bundle reduction theorem, one always gets a Hamiltonian of this form, with the potential being changed by the addition of an amendment to give the *amended potential.* In the case of the pendulum equation, the original Hamiltonian is purely kinetic energy and so the potential term in N, namely $(k_2^2 r^2/2k_3^2)\sin^2\theta$, is entirely amendment. See Volume II for the general theory.

Putting the above discussion together with Exercises 14.9-1 and 14.9-2, one gets

Theorem 15.11.3. *Euler's equations for a free rigid body are Lie–Poisson with the Hamiltonian N for the Lie algebra $\mathbb{R}^3_K$ where the underlying Lie group is the orthogonal group of K if the quadratic form is nondegenerate, and is the Euclidean group of the plane if K has signature* $(+,+,0)$. *In particular, all the groups* $\mathrm{SO}(3), \mathrm{SO}(2,1),$ *and* $\mathrm{SE}(2)$ *occur as the parameters $a, b, c,$ and d are varied. (If the body is a Lagrange body, then the Heisenberg group occurs as well.)*

The same richness of Hamiltonian structure was found in the Maxwell–Bloch system in David and Holm [1992] (see also David, Holm, and Tratnick [1990]). As in the case of the rigid body, the $\mathbb{R}^3$ motion for the Maxwell–Bloch system may also be realized as motion along the intersections of two orthogonally oriented cylinders. However, in this case, one cylinder is parabolic in cross section, while the other is circular. Upon passing to parabolic cylindrical coordinates, the Maxwell–Bloch system reduces to the ideal Duffing equation, while in circular cylindrical coordinates, the pendulum equation results. The $\mathrm{SL}(2, \mathbb{R})$ matrix transformation in the Maxwell–Bloch case provides a parametrized array of (offset) ellipsoids, hyperboloids, and cylinders, along whose intersections the $\mathbb{R}^3$ motion takes place.

Exercises

◇ **Exercise 15.11-1.** Consider the Poisson bracket on $\mathbb{R}^3$ given by

$$\{F_1, F_2\}_K(\Pi) = \nabla K\,(\Pi) \cdot (\nabla F_1(\Pi) \times (\nabla F_2(\Pi))$$

with

$$K(\Pi) = \frac{\Pi_1^2}{2k_1^2} + \frac{\Pi_2^2}{2k_2^2}.$$

Verify that the Poisson bracket on the two-dimensional leaves given by $K =$ constant of this bracket has the expression

$$\{\theta, p\}_{\text{Ellip Cyl}} = \frac{1}{k_1 k_2},$$

where $p = \Pi_3$ and $\theta = \tan\ -1(K_1\Pi_2/k_2\Pi_1)$. What is the symplectic form in these leaves?